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Applying Essentially Non-oscillatory Interpolation to CSEM Modelling

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SUMMARY

Modelling and inversion of Controlled-Source Electro-Magnetics requires accurate interpolation near the resistivity contrasts, where the field component perpendicular to the interface that separates the two resistivities is discontinuous. Simple linear interpolation may produce large errors in that case. Here, we propose to use the essentially non-oscillatory piecewise polynomial interpolation that was designed for piecewise smooth functions that contain discontinuities or have discontinuous first or higher derivatives. The scheme uses a non-linear adaptive algorithm to choose the smoothest stencil. Some examples are presented to illustrate the behaviour of this scheme.
Introduction
Controlled-Source Electro-Magnetics measurements can help to identify hydrocarbon reservoirs. In some cases, data interpretation alone is sufficient as reported, for instance, by Amundsen et al. (2004). In more complex geological settings, forward modelling of electromagnetic diffusion for a range of scenarios can help interpretation, whereas in really complicated cases, full-scale inversion of the data is required.

There exist various modelling codes for electromagnetic modelling of induction current in the earth. Here we will use a fairly efficient code that uses the Finite Integration Technique (Weiland, 1977) for the spatial discretisation and a multigrid scheme as a preconditioner for BiCGSTAB2 (Van der Vorst, 1992; Gutknecht, 1993) to solve the large system of equations (Mulder, 2006; Mulder et al., 2008). Note that large sparse matrix that represents the linear system is actually never formed and memory requirements are therefore limited. The discretisation starts with a grid of block-shaped cells. The three electromagnetic field components are represented as average value on edges, the \(x\)-components on edges parallel to the \(x\)-direction and the \(y\)- and \(z\)-components parallel to their corresponding directions. This approach can be interpreted as a finite-volume generalisation of the scheme by Yee (1966).

Once the electric field components have been computed, we have to sample them at receiver positions. In earlier papers, we applied simple tri-linear interpolation. Near sharp resistivity contrasts, the field component perpendicular the interface that separates the two resistivities is discontinuous. A simple linear interpolation scheme may lead to large errors in that case. Here we apply the essentially non-oscillatory (ENO) scheme of Harten et al. (1997) to reduce such errors. This scheme also provides a correct interpolation when going from edge or cell averages to point values. The ENO scheme was originally proposed for compressible flow computations with shock waves and contact discontinuities and expanded on the earlier breakthrough work of Van Leer (1979).

We will briefly review the ENO scheme and present a number of examples to illustrate the performance of the schemes in electromagnetic applications.

Method
When developing a computer program for modelling 3D CSEM or other kind of problems, we have to discretise the equations on a grid. If we want to sample computed field quantities at specific receiver locations, we need some kind of interpolation procedure. In many cases, it is sufficient to use a polynomial interpolation scheme. The polynomial interpolation methods that are widely used in applications are usually based on a fixed stencil, for example, to get an interpolation result in cell \(i\) with third-order accuracy, we can use information in the three cells \(i - 1, i, \) and \(i + 1\) to construct a local quadratic polynomial, except of course if cell \(i\) is located next to the boundary. This schemes works well for globally smooth solutions. However, when we apply this schemes for functions that have discontinuities, the fixed-stencil interpolation of second- or higher-order accuracy creates numerical oscillations near the discontinuities. This is referred to as the Gibbs phenomenon and is illustrated in Figure 1 on the left side.

The ENO scheme is a high-order interpolation scheme designed for piecewise smooth functions that contain discontinuities or have discontinuous first or higher derivatives. Suppose we are given the cell averages \(u_{i+\frac{1}{2}}\) of a function \(u(x)\) at the centre of each cell,

\[
\bar{u}_{i+\frac{1}{2}} = \frac{1}{\Delta x_i} \int_{x_i}^{x_{i+1}} u(x) \, dx, \quad \Delta x_i = x_{i+1} - x_i,
\]

instead of the (point) function values on the grid. The half index is used to indicate that the value \(\bar{u}_{i+\frac{1}{2}}\) lies at the centre of interval \([x_i, x_{i+1}]\). We reconstruct the function \(u\) with the help of the primitive function \(U(x)\),

\[
U(x) = \int_{-\infty}^{x} u(x') \, dx'.
\]
We use the Newton form to get an interpolation of the function $U$ and take the first derivative of the interpolant to get an interpolation of $u$ itself.

The main idea of the scheme lies at the approximation level where a nonlinear adaptive algorithm is used to automatically choose the locally “smoothest” stencil. When we want to determine a local polynomial interpolation inside cell $i$ with an order of accuracy $r$, we need information from a stencil of $r$ cells around the cell $i$. Clearly, there are exactly $r$ different stencils around the cell $i$ and each stencil will provide an interpolating polynomial. For example, to get a polynomial interpolation for cell $i$ to third-order accuracy, we can use information of the three cells that can be chosen from one of the following stencils: $(i-2, i-1, i)$, $(i-1, i, i+1)$, or $(i, i+1, i+2)$. This freedom is used to assign the cell $i$ a stencil of $r$ cells such that the corresponded polynomial is the smoothest or least oscillatory. Information about the smoothness of the corresponding polynomials is extracted from a table of divided differences, which is defined recursively by

$$D[x_i] = U_i,$$

and

$$D[x_i, \ldots, x_{i+k}] = \frac{D[x_{i+1}, \ldots, x_{i+k}] - D[x_i, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}.$$  

Here, $\{(x_i, U_i := U(x_i))\}$ is the set of data that need to be interpolated. The divided differences themselves can be viewed as approximations of the derivatives of the exact function. How we choose a stencil of $k$ points for which $u$ is smoothest, is basically the same problem as finding the interval where $u$ has the smallest divided differences.

Figure 2 illustrates how the ENO scheme works for 1D interpolation. The scheme starts by setting an initial stencil at interval $[x_i, x_{i+1}]$ (the blue line in the left panel of Figure 2). The function $u(x)$ is approximated by piecewise constant cell averages $\pi_{i+\frac{1}{2}}$. We proceed to obtain a linear interpolant by adding to the above stencil either one cell to the left or one cell to the right (the red line). The choice is based on the absolute values of the divided differences (in the red circle). If $|D[x_{i-1}, x_i, x_{i+1}]| < D[x_i, x_{i+1}, x_{i+2}]$, we select the interval $[x_{i-1}, x_{i+1}]$ as the next stencil, otherwise we choose the interval $[x_i, x_{i+2}]$. This procedure is repeated for larger stencils and higher-degree polynomials, until we obtain a stencil of $r$ cells that corresponds the $r$-th degree smoothest polynomial that interpolates $u$ in the interval $[x_i, x_{i+1}]$.

We end with noting that selection of the smoothest stencil by means of an if-statement may pose problems if the modelling code is used for least-squares inversion of electro-magnetic measurements. In these, a differentiable version can be employed as proposed by Van Albada et al. (1982); see also Mulder and Van Leer (1985).
Figure 2: The ENO scheme adaptively determines a stencil for interpolation by selecting the smallest divided differences. Here, we use $D_i^{(k)} \equiv D[x_i, \ldots, x_i+k]$ for brevity.

**Examples**

As a first example, we test our computer program on a simple model. We choose a piecewise smooth function with isolated discontinuities:

$$y(x) = \sin(x) - \text{sign}(x) - |x - 1| + 1$$

We choose a grid between $-1.5$ and $1.5$ with spacing $0.1$ and compute the cell averages as the input of our program. We also use the cubic spline interpolation. Figure 3 shows the results.

Figure 3: The interpolation using ENO scheme and cubic spline interpolation of a function with isolated discontinuities.

The second example is three-layer problem: air, water, and sediments. These layers have a conductivity $\sigma$ of 0, 3, and 0.5 S/m, respectively, and a relative permittivity $\epsilon_r$ of 1, 80, and 17. Basically, the test is the similar to the three-layer problem in our earlier paper (Mulder et al., 2008). Here, we make a small modification to illustrate the performance of the ENO scheme. We set the water depth at either 200 m or 150 m with a jump at $x = 500$ m from deeper to shallower. A dipole source in the $x$-direction is located at a depth of 175 m and a 750-m distance from the jump in water depth. The frequency is set to 0.5 Hz. An array of receivers is located at a depth of 200 m.

The multigrid code (Mulder, 2006) produces $E_1$ as edge averages. We use the ENO interpolation to replace the edge average by an interpolating curve, that consists of piecewise polynomials (Figure 4). The blue line shows the piecewise linear interpolation, whereas the red line represents the ENO quadratic interpolation. With piecewise linears, we get line segments that cross the average value (the black circles), but for piecewise quadratic curves, the curves may not cross the average value. However, one property of the ENO interpolation is the scheme still has the same edge average in each cell.
Figure 4: Response for the inline electric field component at a depth of 200 m. The figure on the right shows the values along the dashed line in the left panel. Of course, sea bottom receivers can only be found for an inline distance less than 500 m.

Conclusion
We have applied essentially non-oscillatory (ENO) interpolation to obtain accurate values of the electric field components at receiver locations from their computed averages on edges of cells. The ENO scheme properly deals with discontinuities in the solution as well in its derivatives by selecting the smoothest part of the solution.

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References