A unified optical theorem for scalar and vectorial wave fields

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The generalized optical theorem is an integral relation for the angle-dependent scattering amplitude of an inhomogeneous scattering object embedded in a homogeneous background. It has been derived separately for several scalar and vectorial wave phenomena. Here a unified optical theorem is derived that encompasses the separate versions for scalar and vectorial waves. Moreover, this unified theorem also holds for scattering by anisotropic elastic and piezoelectric scatterers as well as bianisotropic (non-reciprocal) EM scatterers. © 2012 Acoustical Society of America. [http://dx.doi.org/10.1121/1.3701880]

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I. INTRODUCTION

The optical theorem finds its origin in the late nineteenth century, when Rayleigh1 and others formulated the relation between the optical refraction index of a scattering object in a homogeneous embedding and its forward scattering amplitude. Later Heisenberg,2 Glauber and Schomaker,3 and others derived a more general theorem for the scattering amplitude in quantum mechanics and other scalar wave phenomena. This theorem, which has become known as the generalized optical theorem, is an integral relation for the scattering amplitude for any angle of incidence and any scattering angle. Both the optical theorem and the generalized optical theorem are a consequence of conservation of energy (or conservation of probability in quantum mechanics). For more extensive reviews, see Newton,4 Marston,5 and Carney et al.6

The generalized optical theorem is most often applied to scalar wave phenomena, but extensions for vectorial wave phenomena have been formulated as well. Snieder7 and Halliday and Curtis8,9 derive an optical theorem for multi-mode elastic surface waves in a layered medium bounded by a free surface. Tan,10 de Hoop,11 and Lu et al.12 discuss the optical theorem for scattering of elastic body waves, and Torrungrueng et al.13 and Lytle et al.14 derive a version for electromagnetic waves.

It has recently been recognized that there is a close connection between the generalized optical theorem and the Green’s function representations5–17 that underlie the methodology of Green’s function retrieval from ambient noise in open systems.18–22 It has been shown that the optical theorem for scalar waves can be derived from the scalar Green’s function representation,23–27 whereas the optical theorems for surface waves and elastic body waves have been derived from elastodynamic Green’s function representations for surface waves8,9 and body waves,12,28 respectively. Halliday and Curtis9 and Douma et al.26 suggested that a unified optical theorem for scalar and vectorial wave fields could possibly be derived from a unified Green’s function representation.29 The aim of this paper is to show that this is indeed the case. Starting with a unified wave equation for scalar and vectorial fields, unified Green’s function representations are derived. Next, following a similar procedure as Douma et al.26 for scalar wave fields, a unified optical theorem for scalar and vectorial wave fields is derived. This unified theorem captures most of the situations discussed above and in addition covers scattering by non-reciprocal materials and piezoelectric materials.

II. RECIPROCITY THEOREMS

The starting point is the following unified wave equation:30–32

$$A\partial_t u + Bu + D_x u = s,$$

in which $$u = u(x, t)$$ is a $$L \times 1$$-vector containing space ($$x$$) and time ($$t$$) dependent wave field quantities, $$A = A(x)$$ and $$B = B(x)$$ are $$L \times L$$ matrices containing space-dependent medium parameters, $$\partial_t$$ denotes differentiation with respect to time, $$D_x$$ is a $$L \times L$$ matrix containing spatial differential operators $$\partial_1, \partial_2, \partial_3,$$ and $$s = s(x, t)$$ is a $$L \times 1$$ source vector. In Appendix A, these vectors and matrices are specified for acoustic waves (for which $$L = 4$$), quantum-mechanical waves ($$L = 4$$), electromagnetic waves in reciprocal and non-reciprocal materials ($$L = 6$$), elastodynamic body waves ($$L = 9$$) and coupled electromagnetic and elastodynamic waves in piezoelectric materials ($$L = 15$$). For all situations, matrix $$D_x$$ obeys the following symmetry relations:

$$D_x = D_x^T,$$

$$D_x = -KD_x K,$$

where superscript $$T$$ denotes transposition and where $$K$$ is a $$L \times L$$ diagonal matrix containing a specific ordering of 1’s and −1’s along the diagonal. Note that $$K$$ obeys the property $$K = K^{-1} = K^T$$. 

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Equation (1) also holds for diffusion phenomena, linearized flow, as well as (coupled) electromagnetic and elastodynamic waves in poroelastic media.31 These cases are not considered here because they do not obey energy conservation and hence there is no optical theorem for these situations.

The temporal Fourier transform of a time-dependent function $f(t)$ is defined as follows:

$$
 f(\omega) = \int_{-\infty}^{\infty} f(t) \exp(i\omega t) dt,
$$

(4)

where $\omega$ is the angular frequency and $i$ the imaginary unit ($i = \sqrt{-1}$). To keep the notation simple, the same symbol is used for time- and frequency-domain functions (here $f$). In the remainder of the main text all functions are in the frequency domain. In the appendixes it is always clear from the context which domain is considered.

In the frequency domain, Eq. (1) becomes

$$
 D_\lambda u = i\omega A u + s,
$$

(5)

where

$$
 A = A(x, \omega), \quad u = u(x, \omega), \quad \text{and} \quad s = s(x, \omega).
$$

The spatial differential operator $D_\lambda$ is the same as in Eq. (1).

What follows is a brief review of the derivation of two unified reciprocity theorems for wave fields obeying the unified wave equation.31 Consider a domain $\mathcal{D}$ enclosed by boundary $\partial \mathcal{D}$ with outward pointing normal vector $n$, see Fig. 1. In this domain there are two independent physical states $\{A_\lambda u_A, s_A\}$ and $\{A_\beta u_\beta, s_\beta\}$, respectively, each state obeying wave equation (5). In Appendix B, the following matrix-vector form of Gauss’s theorem is derived:

$$
 \int_\mathcal{D} \left\{ a^T D_\lambda b + (D_\lambda a)^T b \right\} d^3 x = \int_{\partial \mathcal{D}} a^T N_\lambda n b d^2 x,
$$

(7)

where $a$ and $b$ are arbitrary vector functions and $N_\lambda$ is a $L \times L$ matrix containing the components $n_1, n_2, n_3$ of the normal vector $n$ on $\partial \mathcal{D}$, organized in the same way as $\partial_1, \partial_2, \partial_3$ in matrix $D_\lambda$. Substituting $a = Ku_A$, $b = u_\beta$, and using Eqs. (3) and (5), yields

$$
 \int_\mathcal{D} \left\{ u_\beta^T K s_\beta - s_\beta^T K u_A \right\} d^3 x
$$

$$
 = \int_{\partial \mathcal{D}} u_\beta^T K N_\lambda n u_A d^2 x - i\omega \int_\mathcal{D} u_\beta^T K (A_\lambda - A_\lambda^*) u_A d^3 x,
$$

(8)

where

$$
 A_\lambda^* = K A_\lambda^T K.
$$

(9)

Equation (8) is the unified reciprocity theorem of the convolution type. $A_\lambda^*$ is called the medium parameter matrix of the adjoint medium [which is to be distinguished from the adjoint matrix $A_\lambda^*$ appearing in Eq. (10)]. An adjoint medium is loosely defined as the medium in which, after interchanging a given source and receiver, the same response is obtained as in the original medium before the source and receiver were interchanged. For example, for acoustic waves in a flowing medium, the adjoint medium is the medium with reversed flow.33,34 For all cases discussed in Appendix A, except for electromagnetic waves in bianisotropic materials,35 it holds that $A_\lambda^* = A$, which means that the medium parameters are self-adjoint for these cases. In Sec. III it is confirmed that self-adjointness of the medium parameters is equivalent to the medium being reciprocal. Self-adjointness of the medium parameters is not required for the derivation of the unified optical theorem, see Sec. V.

Next, substitute $a = u_\lambda^*$ and $b = u_\beta$ into Gauss’s theorem (7), where the asterisk (*) denotes complex conjugation. Using Eq. (5), this gives

$$
 \int_\mathcal{D} \left\{ u_\lambda^* s_\lambda + s_\lambda^* u_\beta \right\} d^3 x
$$

$$
 = \int_{\partial \mathcal{D}} u_\lambda^* N_\lambda n u_\beta d^2 x - i\omega \int_\mathcal{D} u_\lambda^* (A_\beta - A_\beta^*) u_\beta d^3 x,
$$

(10)

where the dagger (†) denotes complex conjugation and transposition. Equation (10) is the unified reciprocity theorem of the correlation type. When state $A$ is equal to state $B$, this equation simplifies to

$$
 2\Re \int_\mathcal{D} u^* s d^3 x
$$

$$
 = \int_{\partial \mathcal{D}} u^* N_\lambda n u d^2 x - i\omega \int_\mathcal{D} u^* (A - A^*) u d^3 x,
$$

(11)

where $\Re$ denotes the real part. The left-hand side represents the energy injected into the system by the sources in $\mathcal{D}$. The first integral on the right-hand side is the energy leaving the system through the boundary $\partial \mathcal{D}$ and the second integral on the right-hand side quantifies the energy loss in $\mathcal{D}$. Energy is conserved when $A^* = A$, i.e., when matrix $A$ is self-adjoint (for quantum-mechanical waves, replace “energy” by...
III. GREEN’S FUNCTION REPRESENTATIONS

A Green’s function is defined as the wave field that would be obtained if the source were an impulsive point source \( \delta(x - x') \delta(t) \), or, in the frequency domain, a point source \( \delta(x - x') \) with unit spectrum. Because the source vector \( s \) on the right-hand side of Eq. (8) vanishes, the boundary Green’s wave vectors are combined into a Green’s matrix and the source matrix, according to Eq. (12) for the \( l \)th Green’s wave vector \( g_l \), where the bars denote a reference state, and take for state \( x \) the second integral on the right-hand side of Eq. (8) vanishes. Replacing \( g_l \) by \( i \delta(x - x') \), where \( i_l \) is the \( L \) \( \times \) 1 unit vector \( (0, \ldots, 1, \ldots, 0)^T \), with “1” on the \( l \)th position. Hence, the Green’s wave vector obeys the following equation:

\[
D_l g_l = i \omega A_l g_l + i \delta(x - x'),
\]

where \( g_l = g_l(x, x', \omega) \) is the \( l \)th \( L \times 1 \) Green’s wave vector observed at \( x \), due to a point source of the \( l \)th type at \( x' \). In the following, \( \omega \) is suppressed in the argument list but the coordinate vectors \( x \) and \( x' \) are retained where appropriate. Equation (12) represents a matrix-vector relation for the \( L \) Green’s wave vectors \( g_l \). The \( L \) Green’s vectors are combined into a Green’s matrix and the \( L \) source vectors into a source matrix, according to

\[
(g_1, \ldots, g_l, \ldots, g_L)(x, x') = G(x, x'),
\]

\[
(i_1, \ldots, i_l, \ldots, i_L) \delta(x - x') = I \delta(x - x'),
\]

where \( G(x, x') \) is the \( L \times L \) Green’s wave field matrix and \( I \) is the \( L \times L \) identity matrix. With this notation, Eq. (12) for \( l = 1, \ldots, L \) can be combined into

\[
D_l g = i \omega A_l G + \delta(x - x').
\]

The convolution-type reciprocity theorem (8) is now used to derive the reciprocity properties of the Green’s matrix. To this end, replace \( \{ A_l, u_{A_l}, s_{A_l} \} \) by \( \{ A(x), G(x, x'), I \delta(x - x') \} \) and \( \{ A_l, u_{A_l}, s_{A_l} \} \) by \( \{ A^{(0)}(x), G^{(0)}(x, x'), I \delta(x - x') \} \). Because the medium in state \( B \) is chosen as the adjoint of the medium in state \( A \), the second integral on the right-hand side of Eq. (8) vanishes. Replacing \( D \) by \( \mathbb{R}^3 \) and assuming that outside some sphere with finite radius the medium is homogeneous, isotropic and self-adjoint, the first integral on the right-hand side vanishes as well (Sommerfeld’s radiation conditions). This leaves

\[
\int_{\mathbb{R}^3} \{ \mathcal{G}^{(0)}(x, x') \delta(x - x') \} d^3 x = 0
\]

or

\[
G^{(0)}(x', x'') = KG^T(x'', x')K.
\]

Note that \( G^{(0)}(x', x'') \) is defined in a medium which is the adjoint of the medium in which \( G(x', x') \) is defined. For a self-adjoint medium equation (17) simplifies to

\[
G(x', x'') = KG^T(x'', x')K.
\]

This equation quantifies source-receiver reciprocity. Hence, self-adjointness of the medium is equivalent to the medium being reciprocal.

Next, two unified Green’s function representations are derived. For state \( A \), choose \( \{ A(x), G(x, x'), I \delta(x - x') \} \), where the bars denote a reference state, and take for state \( B \) the actual state, i.e., \( \{ A(x), G(x, x'), I \delta(x - x') \} \). Substitution of these states in the convolution-type and correlation-type reciprocity theorems (8) and (10), respectively, yields [using Eq. (17) for the reference Green’s function]

\[
\chi_D(x')G(x', x'') - \chi_D(x'')G^{(0)}(x', x'') = -\int_{\partial D} \mathcal{G}^{(0)}(x', x)N_x G(x, x'') d^2 x
\]

\[
+ i \omega \int_D \mathcal{G}^{(0)}(x', x)\{ A - \tilde{A}^{(0)} \}(x)G(x, x'') d^3 x
\]

and

\[
\chi_D(x')G(x', x'') + \chi_D(x'')\mathcal{G}^{(0)}(x', x') = \int_D G^T(x', x')N_x G(x, x'') d^2 x
\]

\[
- i \omega \int_D G^T(x', x')\{ A - \tilde{A} \}(x)G(x, x'') d^3 x,
\]

respectively, where \( \chi_D(x') \) is the characteristic function for domain \( D \), defined as

\[
\chi_D(x') = \begin{cases} 1 & \text{for } x' \in D, \\ \frac{1}{2} & \text{for } x' \in \partial D, \\ 0 & \text{for } x' \in \mathbb{R}^3 \setminus \{ D \cup \partial D \}. \end{cases}
\]

The convolution-type Green’s function representation (19) is a basis, for example, for iterative forward modeling of scattered wave fields, using boundary and/or volume integral methods. The correlation-type representation (20) is a basis for the methodology of Green’s function retrieval by cross-correlation of ambient noise in its most general form. A further discussion of these applications is beyond the scope of this paper. Both representations are used in the following sections in the derivation of the unified optical theorem.

IV. INTEGRAL RELATION FOR THE GREEN’S FUNCTION OF THE SCATTERED WAVE FIELD

The generalized optical theorem is an integral relation for the angle-dependent scattering amplitude of a scattering object. Here an integral relation for the Green’s function of the scattered wave field is derived, which will be used as the basis for the derivation of the generalized optical theorem in the next section.

The total Green’s function \( G(x, x') \) in the actual medium \( A(x) \) is the sum of the reference Green’s function \( \tilde{G}(x, x') \)

in the reference medium $\tilde{A}(x)$ and the Green’s function $G^i(x, x')$ of the scattered wave field, hence

$$G(x, x') = \tilde{G}(x, x') + G^i(x, x'). \tag{22}$$

In Sec. V the reference medium will be taken homogeneous, isotropic, reciprocal, and lossless, but for the moment the choice of the reference medium is arbitrary. The correlation-type representation (20) will now be used to find an expression for the following integral:

$$\int_{\mathcal{D}} \{G^i(x, x')\}^\dagger N_s G^i(x, x'')dx', \tag{23}$$

with $x'$ and $x''$ both in $\mathcal{D}$. A compact notation to represent integrals of this form is

$$\mathcal{L}(G_1, G_2) = \int_{\mathcal{D}} G^1(x, x') N_s G^2(x, x'') dx'. \tag{24}$$

Here the subscripts 1 and 2 at the left-hand side correspond to the source positions $x'$ and $x''$, respectively, of the two Green’s functions. Substitution of Eq. (22) into Eq. (24) yields

$$\mathcal{L}(G_1, G_2) = \mathcal{L}(G_1, G_2) + \mathcal{L}(G_1, G_2') + \mathcal{L}(G_1', G_2) + \mathcal{L}(G_1', G_2'). \tag{25}$$

Using Eq. (22) again, the second and third term in the right-hand side of Eq. (25) can be expressed as

$$\mathcal{L}(G_1, G_2') = \mathcal{L}(G_1, G_2) - \mathcal{L}(G_1, G_2), \tag{26}$$

$$\mathcal{L}(G_1', G_2) = \mathcal{L}(G_1, G_2) - \mathcal{L}(G_1, G_2). \tag{27}$$

Substituting this into Eq. (25) and bringing the last term to the left-hand side gives

$$\mathcal{L}(G_1', G_2') = \mathcal{L}(G_1, G_2) + \mathcal{L}(G_1, G_2') - \mathcal{L}(G_1, G_2) \tag{28}$$

Note that the left-hand side is the sought integral of Eq. (23), which has now been expressed in terms of integrals containing the total and the reference Green’s functions. The right-hand side is evaluated term by term. Taking the reference medium equal to the actual medium in Eq. (20), and using the fact that $x'$ and $x''$ are both situated in $\mathcal{D}$, yields for the first term on the right-hand side of Eq. (28)

$$\mathcal{L}(G_1, G_2) = G(x', x'') + G^i(x', x') \tag{29}$$

$$+ io \int_{\mathcal{D}} G^1(x, x') \{\mathcal{A} - \mathcal{A}^\dagger\} (x)G(x, x'')dx'. \tag{30}$$

The third term on the right-hand side of Eq. (28) follows directly from Eq. (20), hence

$$\mathcal{L}(G_1, G_2') = G(x', x'') + G^i(x', x') \tag{31}$$

$$+ io \int_{\mathcal{D}} G^1(x, x') \{\mathcal{A} - \mathcal{A}^\dagger\} (x)G(x, x'')dx'. \tag{32}$$

Finally, interchanging the roles of the total and reference Green’s functions, yields for the fourth term

$$\mathcal{L}(G_1', G_2) = G(x', x'') + G^i(x', x') \tag{33}$$

$$- io \int_{\mathcal{D}} G^1(x, x') \{\mathcal{A} - \mathcal{A}^\dagger\} (x)G(x, x'')dx'. \tag{34}$$

Substituting Eqs. (29)–(32) into the right-hand side of Eq. (28) and replacing the left-hand side by expression (23) gives

$$\int_{\mathcal{D}} \{G^i(x, x')\}^\dagger N_s G^i(x, x'')dx'$$

$$= + io \int_{\mathcal{D}} G^1(x, x') \{\mathcal{A} - \mathcal{A}^\dagger\} (x)G(x, x'')dx'$$

$$+ io \int_{\mathcal{D}} G^1(x, x') \{\mathcal{A} - \mathcal{A}^\dagger\} (x)G(x, x'')dx'$$

$$- io \int_{\mathcal{D}} G^1(x, x') \{\mathcal{A} - \mathcal{A}^\dagger\} (x)G(x, x'')dx'$$

$$\int_{\mathcal{D}} \{G^i(x, x')\}^\dagger N_s G^i(x, x'')dx'. \tag{35}$$

V. THE UNIFIED OPTICAL THEOREM

From here onward, consider a small scattering domain $\mathcal{D}$ around the origin, embedded in a reference domain $\mathbb{R}^3$, see Fig. 2. The scattering domain may be arbitrary inhomogeneous, anisotropic, and non-reciprocal, but it is assumed to be lossless, hence $\mathcal{A}(x) = \mathcal{A}^\dagger(x)$. The reference state is taken homogeneous, isotropic, reciprocal, and lossless, hence $\mathcal{A} = \mathcal{A}^{\text{ref}} = \mathcal{A}^\dagger$. Outside the scattering domain $\mathcal{D}$, centered at the origin, it holds that $\mathcal{A}(x) = \mathcal{A}$. For $\partial \mathcal{D}$, choose a large spherical boundary, centered at the origin. Define a unit vector $\hat{x}$ in the direction of $x$, according to $x = \hat{x}/|\hat{x}|$. Hence, the normal $n$ on $\partial \mathcal{D}$ equals $\hat{x}$, for $x$ on $\partial \mathcal{D}$. Using all this in Eq. (33) yields

$$\int_{\partial \mathcal{D}} \{G^i(x, x')\}^\dagger M(x)G^i(x, x'')dx$$

$$= - io \int_{\partial \mathcal{D}} G^1(x, x') \{\mathcal{A}(x) - \mathcal{A}\} G(x, x'')dx'$$

$$+ io \int_{\partial \mathcal{D}} G^1(x, x') \{\mathcal{A}(x) - \mathcal{A}\} G(x, x'')dx'. \tag{36}$$
with \( \mathbf{M}(\hat{x}) \) defined as \( \mathbf{N}_s \), but with all \( n_i \) replaced by \( \hat{x}_i \). Express the far field of the Green’s function of the scattered wave field as

\[
G'(\mathbf{x}, \mathbf{x}') = i\zeta \mathbf{G}(\mathbf{x}, 0) \mathbf{F}(\hat{x}, -\hat{x}') \mathbf{G}(0, \mathbf{x}') \tag{35}
\]

and a similar expression for \( G''(\mathbf{x}, \mathbf{x}'') \), with \( \mathbf{x} \) on \( \partial \mathbb{D} \), and \( \mathbf{x}', \mathbf{x}'' \) in \( \mathbb{D} \), all far from the scattering domain \( \mathbb{D}_s \), see Fig. 2. Here \( \mathbf{F}(\hat{x}, -\hat{x}') \) is a \( L \times L \) matrix containing angle-dependent scattering amplitudes. Similar to \( \hat{x} \), vectors \( \mathbf{x}' \) and \( \mathbf{x}'' \) are unit vectors in the direction of \( \mathbf{x}' \) and \( \mathbf{x}'' \), respectively. Finally, \( i\zeta \) is a conveniently chosen normalization factor that compensates for factors in the reference Green’s function, see Appendix C for details. Next, the optical theorem for the scattering matrix \( \mathbf{F}(\hat{x}, -\hat{x}') \) is derived.

Step 1: substitution of Eq. (35) and a similar expression for \( G''(\mathbf{x}, \mathbf{x}'') \) into the left-hand side (LHS) of Eq. (34) gives

\[
\text{LHS of Eq. (34)} = i\zeta \mathbf{G}(\mathbf{x}, 0) \mathbf{F}(\hat{x}, -\hat{x}') \mathbf{G}(0, \mathbf{x}')
\times \mathbf{M}(\hat{x}) \mathbf{G}(\mathbf{x}, 0) \mathbf{F}(\hat{x}, -\hat{x}'') \mathbf{G}(0, \mathbf{x}''). \tag{36}
\]

In Appendix C it is shown that

\[
\mathbf{G}^\dagger(\mathbf{x}, 0) \mathbf{M}(\hat{x}) \mathbf{G}(\mathbf{x}, 0) = \frac{2}{\zeta} \Theta(\hat{x}) |\hat{x}|^2, \tag{37}
\]

where \( \Theta(\hat{x}) \) is a function of the unit vector \( \hat{x} \) and the parameters of the embedding medium. Hence

\[
\text{LHS of Eq. (34)} = 2 \zeta \mathbf{G}^\dagger(\mathbf{x}, 0) \mathbf{F}(\hat{x}, -\hat{x}') \Theta(\hat{x})
\times \mathbf{F}(\hat{x}, -\hat{x}'') d\Omega_\mathbf{x} \mathbf{G}(0, \mathbf{x}''), \tag{38}
\]

with \( d\Omega_\mathbf{x} = d^2\mathbf{x}/|\mathbf{x}|^2 \).

Step 2: Eq. (19) is used to derive an explicit expression for the scattering matrix \( \mathbf{F} \). Because \( \mathbf{A}^{(s)} = \mathbf{A} \) in the reference state, it holds that \( \mathbf{G}^{(s)} = \tilde{\mathbf{G}} \). Hence, taking into account that \( \mathbf{x}' \) and \( \mathbf{x}'' \) are situated in \( \mathbb{D}_s \), the left-hand side of Eq. (19) is equal to the Green’s function \( \mathbf{G}'(\mathbf{x}', \mathbf{x}'') \) for the scattered wave field. Because outside \( \partial \mathbb{D} \) the parameters of the reference state as well as of the actual state are homogeneous, isotropic, reciprocal and lossless, the boundary integral on the right-hand side of Eq. (19) vanishes on account of Sommerfeld’s radiation conditions. This leaves

\[
\mathbf{G}'(\mathbf{x}', \mathbf{x}'') = i\omega \int_{\partial \mathbb{D}_s} \mathbf{G}(\mathbf{x}', \mathbf{x}) \{ \mathbf{A}(\mathbf{x}) - \tilde{\mathbf{A}} \} \mathbf{G}(\mathbf{x}, \mathbf{x}'') d^3\mathbf{x}. \tag{39}
\]

For all \( \mathbf{x} \) in the integration domain \( \partial \mathbb{D}_s \) it holds that \( |\mathbf{x}| \ll |\mathbf{x}'| \) and \( |\mathbf{x}| \ll |\mathbf{x}'| \), see Fig. 3. Approximate \( \mathbf{G}(\mathbf{x}, \mathbf{x}') \) by

\[
\tilde{\mathbf{G}}(\mathbf{x}, \mathbf{x}') = \mathbf{P}(\mathbf{x}, \mathbf{x}') \tilde{\mathbf{G}}(\mathbf{0}, \mathbf{x}'), \quad |\mathbf{x}| \ll |\mathbf{x}'|, \tag{40}
\]

where \( \mathbf{P}(\mathbf{x}, \mathbf{x}') \) is a matrix containing plane-wave functions, see Appendix C for details. Similarly,

\[
\mathbf{G}(\mathbf{x}, \mathbf{x}'') = \mathbf{P}(\mathbf{x}, \mathbf{x}'') \tilde{\mathbf{G}}(\mathbf{0}, \mathbf{x}''), \quad |\mathbf{x}| \ll |\mathbf{x}'|, \tag{41}
\]

Applying symmetry relation (18) for the reference Green’s function, yields

\[
\mathbf{P}(\mathbf{x}, \mathbf{x}) = \mathbf{K} \mathbf{P}(\mathbf{x}, \mathbf{x}') \mathbf{K}. \tag{42}
\]

Approximate \( \mathbf{G}(\mathbf{x}, \mathbf{x}'') \) by

\[
\mathbf{G}(\mathbf{x}, \mathbf{x}'') = \mathbf{P}(\mathbf{x}, \mathbf{x}'') \tilde{\mathbf{G}}(\mathbf{0}, \mathbf{x}''), \quad |\mathbf{x}| \ll |\mathbf{x}'|, \tag{43}
\]
The integrals in this equation resemble that in Eq. (44),

\[ F(\mathbf{x}', -\mathbf{x}'') = \frac{\omega}{k} \int_{\mathbb{D}_s} \mathbf{P}(\mathbf{x}', \mathbf{x}) \{ \mathbf{A}(\mathbf{x}) - \mathbf{A} \} \mathbf{P}(\mathbf{x}, \mathbf{x}'') d^3x. \]

(44)

Step 3: substituting Eqs. (40) and (43) into the right-hand side (RHS) of Eq. (34) gives

\[
\text{RHS of (34)} = -i\omega \int_{\mathbb{D}_s} \mathbf{P}(\mathbf{x}, \mathbf{x}') \{ \mathbf{A}(\mathbf{x}) - \mathbf{A} \} \mathbf{P}(\mathbf{x}, \mathbf{x}'') d^3x

+ i\omega \int_{\mathbb{D}_s} \mathbf{P}(\mathbf{x}, \mathbf{x}') \{ \mathbf{A}(\mathbf{x}) - \mathbf{A} \} \mathbf{P}(\mathbf{x}, \mathbf{x}'') d^3x \]

\[ G(0, 0'') = 0(\mathbf{x}) G(\mathbf{x}, 0) \theta(\mathbf{r}). \]

(51)

According to Appendix C, for acoustic, quantum-mechanical, and electromagnetic waves, the reference Green’s functions in Eq. (51) can be written as

\[ G(0, 0') = \theta(\mathbf{r}) G(\mathbf{x}, 0) \theta(\mathbf{r}), \]

(52)

\[ G(0, 0') = \theta(\mathbf{r}) G(\mathbf{x}, 0) \theta(\mathbf{r}), \]

(53)

where \( \theta(\mathbf{r}) \) is a function of the unit vector \( \mathbf{r} \) and the parameters of the embedding medium. For acoustic and quantum-mechanical waves \( G(\mathbf{x}, 0) \) is actually a scalar function, i.e., \( G(\mathbf{x}, 0) \), whereas for electromagnetic waves \( G(\mathbf{x}, 0) \) is a \( 3 \times 3 \) matrix. Substituting Eqs. (52) and (53) into Eq. (51) yields

\[ G'(\mathbf{x}, \mathbf{x}') = \theta(\mathbf{r}) G'(\mathbf{x}, \mathbf{x}') \theta(\mathbf{r}), \]

(54)

with

\[ f(\mathbf{x}, -\mathbf{x}') = \theta(\mathbf{r}) f(\mathbf{x}, -\mathbf{x}') \theta(\mathbf{r}). \]

(56)

Note that Eq. (55) has the same form as Eq. (51), except that in Eq. (55) all functions are scalars (for acoustic and quantum-mechanical waves) or \( 3 \times 3 \) matrices (for electromagnetic waves). Apply \( \theta(\mathbf{r}) \) and \( \theta(\mathbf{r}) \) to both sides of the unified optical theorem [Eq. (50)], as follows:

\[
\theta(\mathbf{r}) \frac{k}{4\pi} \text{d}\Omega_k
\]

(57)

or, renaming \( -\mathbf{x}' \) and \( -\mathbf{x}" \) as \( \mathbf{x}' \) and \( \mathbf{x}" \), respectively,

\[
\Theta(\mathbf{r}) = \frac{k}{4\pi} \theta(\mathbf{r}) \theta(\mathbf{r}).
\]

(58)

This is the well-known generalized optical theorem for scalar waves. Usually it is assumed that the scattering domain \( \mathbb{D}_s \) is characterized by a single parameter (e.g., a refraction-index contrast or a scattering potential). The derivation that led to Eq. (50) accounts for two contrast
parameters. This can be seen as follows. The scattering function \( f \) is expressed in terms of the \( 4 \times 4 \) matrix \( \mathbf{F} \) via Eq. (56), which is related to the contrast matrix \( \mathbf{A}(x) - \tilde{\mathbf{A}} \) in domain \( \mathbb{D}_0 \) via Eq. (44). For the acoustic situation this contrast matrix contains, via Eq. (A7) (with \( b^P = b^S = 0 \)), the compressibility and mass density contrasts. Douma et al. 26 also derived Eq. (59) for a scattering domain with two parameter contrasts, using the same method that is here extended using a unified notation.

For electromagnetic waves, matrix \( \Theta(\mathbf{x}) \) is given by
\[
\Theta(\mathbf{x}) = \frac{\mu_k}{4\pi} \delta(\mathbf{x})(1 - \Gamma(\mathbf{x})) \theta^T(\mathbf{x}). \tag{60}
\]

Following the same procedure as above yields
\[
\frac{\mu_k}{4\pi} \int \delta^T(\mathbf{x}, \mathbf{x}')(1 - \Gamma(\mathbf{x})) f(\mathbf{x}, \mathbf{x}') d\Omega_k
= \frac{1}{2i} \left( f(\mathbf{x}', \mathbf{x}'') - f(\mathbf{x}'', \mathbf{x}') \right), \tag{61}
\]
with
\[
\Gamma(\mathbf{x}) = \begin{pmatrix}
\tilde{x}_1^2 & \tilde{x}_1 \tilde{x}_2 & \tilde{x}_1 \tilde{x}_3 \\
\tilde{x}_2 \tilde{x}_1 & \tilde{x}_2^2 & \tilde{x}_2 \tilde{x}_3 \\
\tilde{x}_3 \tilde{x}_1 & \tilde{x}_3 \tilde{x}_2 & \tilde{x}_3^2
\end{pmatrix}. \tag{62}
\]

Equation (61) is the generalized optical theorem for electromagnetic waves.13,14 In its present compact form it holds for a scattering domain with arbitrary inhomogeneous, anisotropic, and possibly non-reciprocal parameters contained in matrix \( \mathbf{A}(x) - \tilde{\mathbf{A}} \), with \( \tilde{\mathbf{A}} \) defined in Eq. (A22).

For the elastodynamic situation, \( \mathbf{G}(\mathbf{x}, \mathbf{0}) \) and \( \tilde{\mathbf{G}}(\mathbf{0}, \mathbf{x}') \) in Eq. (51) are defined as
\[
\mathbf{G}(\mathbf{x}, \mathbf{0}) = \theta_P(\mathbf{x}) \mathbf{G}_P(\mathbf{x}) \theta^T_P(\mathbf{x}) + \theta_S(\mathbf{x}) \tilde{\mathbf{G}}_S(\mathbf{x}) \theta^T_S(\mathbf{x}), \tag{63}
\]
\[
\tilde{\mathbf{G}}(\mathbf{0}, \mathbf{x}') = \theta_P(\mathbf{x}) \mathbf{G}_P(\mathbf{x}) \theta^T_P(\mathbf{x}')
+ \theta_S(\mathbf{x}) \tilde{\mathbf{G}}_S(\mathbf{x}) \theta^T_S(\mathbf{x}'), \tag{64}
\]
where \( \mathbf{G}_P(\mathbf{x}) \) and \( \tilde{\mathbf{G}}_S(\mathbf{x}) \) are \( 3 \times 3 \) Green’s matrices for \( P \)- and \( S \)-waves, respectively [Eqs. (C37) and (C38)]. Substituting Eqs. (63) and (64) into Eq. (51) gives
\[
\mathbf{G}'(\mathbf{x}, \mathbf{x}') = \theta_P(\mathbf{x}) \mathbf{G}_P(\mathbf{x}, \mathbf{x}') \theta^T_P(\mathbf{x}'),
+ \theta_S(\mathbf{x}) \mathbf{G}_S(\mathbf{x}, \mathbf{x}') \theta^T_S(\mathbf{x}'), \tag{65}
\]
with
\[
\mathbf{G}'_{Q,R}(\mathbf{x}, \mathbf{x}') = i\zeta \mathbf{G}_Q(\mathbf{x}) f_{Q,R}(\mathbf{x}, -\mathbf{x}') \mathbf{G}_R(-\mathbf{x}'), \tag{66}
\]
and
\[
f_{Q,R}(\mathbf{x}, -\mathbf{x}') = \theta^T_Q(\mathbf{x}) \mathbf{F}(\mathbf{x}, -\mathbf{x}') \theta_R(-\mathbf{x}'), \tag{67}
\]
where each of the subscripts \( Q \) and \( R \) can stand for either \( P \) or \( S \). Here \( f_{Q,R}(\mathbf{x}, -\mathbf{x}') \) is a \( 3 \times 3 \) scattering matrix for an incident \( R \)-type wave in the \(-\mathbf{x}'\) direction, scattered as a \( Q \)-type wave in the \( \mathbf{x} \) direction.

Apply \( \theta^T_Q(\mathbf{x}') \) to both sides of the unified optical theorem [Eq. (50)], as follows:
\[
\int \theta^T_Q(\mathbf{x}')(\mathbf{F}(\mathbf{x}, \mathbf{x}'') - \mathbf{F}(\mathbf{x}', \mathbf{x}')) \theta_R(\mathbf{x}'') d\Omega_k
= \frac{1}{2i} \Theta^T_Q(\mathbf{x}') \mathbf{F}(\mathbf{x}, \mathbf{x}'') \Theta(\mathbf{x}) d\Omega_k. \tag{68}
\]

According to Appendix C matrix \( \Theta(\mathbf{x}) \) is given by
\[
\Theta(\mathbf{x}) = \frac{\omega}{4\pi_p c_p} \left( \frac{1}{c_p^3} \mathbf{P}_F(\mathbf{x}) \Gamma(\mathbf{x}) \mathbf{P}_F^T(\mathbf{x})
+ \frac{1}{c_p^3} \mathbf{P}_Q \mathbf{P}_Q^T \mathbf{F}(\mathbf{x}, \mathbf{x}'') - \mathbf{F}(\mathbf{x}', \mathbf{x}'') \right) \tag{69}
\]
Substituting this into Eq. (68), using Eq. (67), taking into account that \( \mathbf{P}_F(\mathbf{x}) \) and \( \mathbf{P}_Q(\mathbf{x}) \) are real-valued, yields
\[
\int \Theta^T_Q(\mathbf{x}') \mathbf{F}(\mathbf{x}, \mathbf{x}'') \Theta(\mathbf{x}) d\Omega_k
= \frac{\omega}{4\pi_p c_p} \left( \int \mathbf{P}_F(\mathbf{x}) \Gamma(\mathbf{x}) \mathbf{P}_F^T(\mathbf{x})
+ \frac{1}{c_p^3} \mathbf{P}_Q \mathbf{P}_Q^T \mathbf{F}(\mathbf{x}, \mathbf{x}'') - \mathbf{F}(\mathbf{x}', \mathbf{x}'') \right) \tag{70}
\]
with \( \Gamma(\mathbf{x}) \) again defined in Eq. (62). Equation (70) is the generalized optical theorem for elastodynamic \( P \)- and \( S \)-waves.12

For a piezoelectric scattering domain \( \mathbb{D}_E \), the scattering matrix \( \mathbf{F}(\mathbf{x}, -\mathbf{x}') \) is subdivided as follows:
\[
\mathbf{F}(\mathbf{x}, -\mathbf{x}') = \begin{pmatrix}
\mathbf{P}^{EM,EM}(\mathbf{x}, -\mathbf{x}') & \mathbf{F}^{EM,ED}(\mathbf{x}, -\mathbf{x}') \\
\mathbf{P}^{ED,EM}(\mathbf{x}, -\mathbf{x}') & \mathbf{F}^{ED,ED}(\mathbf{x}, -\mathbf{x}')
\end{pmatrix}, \tag{71}
\]
where superscripts \( EM \) and \( ED \) stand for electromagnetic and elastodynamic waves, respectively. The second superscript refers to the type of incident wave, propagating in the \(-\mathbf{x}'\) direction, whereas the first superscript refers to the type of scattered wave, propagating in the \( \mathbf{x} \) direction. Substitute this expression into the unified optical theorem [Eq. (50)], together with Eq. (C68) for \( \Theta(\mathbf{x}) \), and rewrite the result in terms of its submatrices. This yields
\[
\int \left\{ \mathbf{F}^{EM,EM}(\mathbf{x}, -\mathbf{x}') \right\}^T \Theta^{EM}(\mathbf{x}) \mathbf{F}^{V,EM}(\mathbf{x}, \mathbf{x}'') d\Omega_k
+ \int \left\{ \mathbf{F}^{ED,EM}(\mathbf{x}, -\mathbf{x}') \right\}^T \Theta^{ED}(\mathbf{x}) \mathbf{F}^{V,ED}(\mathbf{x}, \mathbf{x}'') d\Omega_k
= \frac{1}{2i} \left( \mathbf{F}^{V,EM}(\mathbf{x}', \mathbf{x}'') - \mathbf{F}^{V,ED}(\mathbf{x}', \mathbf{x}'') \right), \tag{72}
\]
where each of the superscripts \( U \) and \( V \) can stand for either \( EM \) or \( ED \). Here \( \Theta^{EM}(\mathbf{x}) \) and \( \Theta^{ED}(\mathbf{x}) \) are defined by Eqs. (60) and (69), respectively. Introduce a vector \( \theta^T_Q(\mathbf{x}') \). For \( U = EM \) this is the same as vector \( \theta(\mathbf{x}') \) used for electromagnetic waves.
[e.g., as in Eq. (60)]; in this case subscript $Q$ is a dummy subscript. For $U = ED$ this vector is the same as $\theta_Q(x')$ used for elastodynamic waves [e.g., as in Eq. (69)]; in this case subscript $Q$ can stand for either $P$ or $S$. Apply $\left\{\theta_Q^U(x')\right\}^T$ and $\theta_Q^V(x')$ to both sides of Eq. (72), in a similar way as in Eq. (68), and substitute Eqs. (60) and (69). This gives

$$
\frac{(\mu k)^{\text{EM}}}{4\pi} \left\{\frac{\text{EM} U}{Q} \left(\hat{x}, \hat{x}'\right)\right\}^T \left[I - \hat{\Gamma}(\hat{x})\right]^{\text{EM} V} \left(\hat{x}, \hat{x}''\right) d\Omega_k
+ \frac{\omega}{4\pi\rho c_p} \left\{\left[I - \hat{\Gamma}(\hat{x})\right]^{\text{EM} U} \left(\hat{x}, \hat{x}'\right)\right\}^T \hat{\Gamma}^{\text{EM} V} \left(\hat{x}, \hat{x}''\right) d\Omega_k
+ \frac{\omega}{4\pi\rho c_s} \left\{\left[I - \hat{\Gamma}(\hat{x})\right]^{\text{EM} U} \left(\hat{x}, \hat{x}'\right)\right\}^T \hat{\Gamma}^{\text{EM} V} \left(\hat{x}, \hat{x}''\right) d\Omega_k
= \frac{1}{2i} \left\{\left[I - \hat{\Gamma}(\hat{x})\right]^{\text{EM} U} \left(\hat{x}, \hat{x}'\right) - \left[I - \hat{\Gamma}(\hat{x})\right]^{\text{EM} V} \left(\hat{x}, \hat{x}''\right)\right\},
$$

(73)

with

$$
\hat{\Gamma}_{Q,R}^{U,V}(\hat{x}', \hat{x}'') = \left\{\theta_Q^U(x')\right\}^T \hat{\Gamma}^{U,V}(\hat{x}', \hat{x}'') \theta_R^V(x'').
$$

(74)

Equation (73) is the generalized optical theorem for electromagnetic and elastodynamic $P$- and $S$-waves, scattered by a piezoelectric contrast in a homogeneous, isotropic embedding.

VII. CONCLUSIONS

Recently, Douma et al.\textsuperscript{26} derived the generalized optical theorem from reciprocity theorems for acoustic waves in perturbed media. They suggested that their approach could possibly be used to derive a unified optical theorem from a unified Green’s function representation.\textsuperscript{29} Here it has been shown that this can indeed be done. Equation (50) formulates the unified optical theorem in a compact way. It has been shown in Sec. VI that Eq. (50) encompasses most versions of the optical theorem that have been presented in the literature. Moreover, this unified optical theorem also holds for scattering by anisotropic elastic and piezoelectric scatterers and by bianisotropic (i.e., non-reciprocal) electromagnetic scatterers.

Among the applications of the generalized optical theorem mentioned in the literature are (1) testing numerical modeling schemes for scattering amplitudes,\textsuperscript{5} (2) reconstructing the structure of a scatterer from power extinction experiments,\textsuperscript{40} and (3) retrieving the scattered part of the Green’s function from ambient noise and explaining the spurious events that occur when the noise is not equipartitioned.\textsuperscript{23} The unified optical theorem formulated in Eq. (50) provides a starting point for applying these and other methods to the different types of scatterers handled in this paper.

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APPENDIX A: MATRIX-VECTOR WAVE EQUATIONS

1. Acoustic wave equation

The basic equations for acoustic wave propagation in an inhomogeneous, dissipative, non-flowing fluid are the linearized equation of motion

$$
\rho \partial_t v_i + b^i v_i + \partial_i p = f_i
$$

(1A)

and the linearized stress-strain relation

$$
\kappa \partial_t p + b^i p + \partial_i v_i = q.
$$

(2A)

Lower-case latin subscripts (except $t$) take on the values 1, 2, and 3 and Einstein’s summation convention applies to repeated indices. Here $p = p(x, t)$ and $v_i = v(x, t)$ represent the acoustic wave field in terms of acoustic pressure and particle velocity, respectively; $\rho = \rho(x)$ and $\kappa = \kappa(x)$ are the medium parameters mass density and compressibility, respectively; $b^i = b^i(x)$ and $b^i = b^i(x)$ are the loss parameters of the medium; finally, $f_i = f_i(x, t)$ and $q = q(x, t)$ represent the sources in terms of external volume force and volume injection rate, respectively. These equations can be combined into the general matrix-vector wave Eq. (1), with

$$
u = \begin{pmatrix} p \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad s = \begin{pmatrix} q \\ f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad A = \begin{pmatrix} \kappa & 0 & 0 & 0 \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & \rho \end{pmatrix},
$$

(3A)

$$
B = \begin{pmatrix} b^1 & 0 & 0 & 0 \\ 0 & b^2 & 0 & 0 \\ 0 & 0 & b^3 & 0 \\ 0 & 0 & 0 & b^3 \end{pmatrix}, \quad D_x = \begin{pmatrix} \partial_t & \partial_1 & \partial_2 & \partial_3 \\ \partial_1 & 0 & 0 & 0 \\ \partial_2 & 0 & 0 & 0 \\ \partial_3 & 0 & 0 & 0 \end{pmatrix}.
$$

(4A)

Note that $D_x$ obeys symmetry relations (2) and (3), with $K$ defined as

$$
K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
$$

(5A)

Matrices $N_x$, and $M(\hat{x})$, introduced in Eqs. (7) and (34), respectively, are defined as

$$
N_x = \begin{pmatrix} 0 & n_1 & n_2 & n_3 \\ n_1 & 0 & 0 & 0 \\ n_2 & 0 & 0 & 0 \\ n_3 & 0 & 0 & 0 \end{pmatrix}, \quad M(\hat{x}) = \begin{pmatrix} 0 & \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ \hat{x}_1 & 0 & 0 & 0 \\ \hat{x}_2 & 0 & 0 & 0 \\ \hat{x}_3 & 0 & 0 & 0 \end{pmatrix}.
$$

(6A)

The frequency-domain matrix $A$, defined in Eq. (6), is given by

$$
A(x, \omega) = \begin{pmatrix} \kappa(x, \omega) & 0 & 0 & 0 \\ 0 & \rho(x, \omega) & 0 & 0 \\ 0 & 0 & \rho(x, \omega) & 0 \\ 0 & 0 & 0 & \rho(x, \omega) \end{pmatrix}.
$$

(7A)
with
\[ \kappa(x, \omega) = \kappa(x) - \frac{b^f(x)}{i\omega}, \] \hspace{0.5cm} (A8) \]
\[ \rho(x, \omega) = \rho(x) - \frac{b^f(x)}{i\omega}. \] \hspace{0.5cm} (A9)

Note that \( \mathbf{A} = \mathbf{K} \mathbf{A}^T \mathbf{K} \). Combined with Eq. (9) this implies \( \mathbf{A}^{(a)} = \mathbf{A} \), meaning that the medium is reciprocal. Energy is conserved when \( \mathbf{A}^T = \mathbf{A} \), i.e., when \( \Im \{\kappa(x, \omega)\} = \Im \{\rho(x, \omega)\} = 0 \), where \( \Im \) denotes the imaginary part.

2. Quantum-mechanical wave equation

Schrödinger’s wave equation for a particle with mass \( m \) in a potential \( V = V(x) \) is given by\(^{41,42} \)
\[ i\hbar \partial_t \psi = -\frac{\hbar^2}{2m} \partial_x^2 \psi + V \psi, \] \hspace{0.5cm} (A10)
where \( \psi = \psi(x, t) \) is the wave function and \( \hbar = h/2\pi \), with \( h \) Planck’s constant. This equation can be captured in the general matrix-vector wave Eq. (1), with
\[
\mathbf{u} = \begin{pmatrix} \psi \\ \psi' \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 2m & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i/\hbar \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2mV/\hbar & 0 & 0 & 0 \\ 0 & -i/\hbar & 0 & 0 \\ 0 & 0 & -i/\hbar & 0 \\ 0 & 0 & 0 & -i/\hbar \end{pmatrix}, \] \hspace{0.5cm} (A11)
and \( \mathbf{s} \) a null-vector. Furthermore, \( \mathbf{D}_x, \mathbf{K}, \mathbf{N}_x, \) and \( \mathbf{M}(\mathbf{x}) \) are defined in Eqs. (A4)–(A6). The frequency-domain matrix \( \mathbf{A} \), defined in Eq. (6), is given by
\[ \mathbf{A}(x, \omega) = \begin{pmatrix} 2m \left( 1 - \frac{V(x)}{\hbar \omega} \right) & 0 & 0 & 0 \\ 0 & 1/\hbar \omega & 0 & 0 \\ 0 & 0 & 1/\hbar \omega & 0 \\ 0 & 0 & 0 & 1/\hbar \omega \end{pmatrix}. \] \hspace{0.5cm} (A12)

Note that \( \mathbf{A} = \mathbf{K} \mathbf{A}^T \mathbf{K} \), hence \( \mathbf{A}^{(a)} = \mathbf{A} \), meaning that reciprocity is obeyed. Furthermore, \( \mathbf{A}^T = \mathbf{A} \), hence, probability is conserved.

3. Electromagnetic wave equation

Maxwell’s equations for electromagnetic wave propagation read\(^{38,43} \)
\[ \varepsilon_{ik} \partial_t E_k + \sigma_{ik} E_k - \varepsilon_{ijk} \partial_j H_k = -J^e_k, \] \hspace{0.5cm} (A13)
\[ \mu_{km} \partial_t H_m + \varepsilon_{kmn} \partial_n E_m = -J^m_k, \] \hspace{0.5cm} (A14)
where \( E_k = E_k(x, t) \) and \( H_k = H_k(x, t) \) are the electric and magnetic field strengths, respectively; \( \varepsilon_{ik} = \varepsilon_{ik}(x) \), \( \mu_{km} = \mu_{km}(x) \), and \( \sigma_{ik} = \sigma_{ik}(x) \) are the anisotropic permittivity, permeability, and conductivity, respectively; \( J^e_k = J^e_k(x, t) \) and \( J^m_k = J^m_k(x, t) \) are source functions in terms of the external electric and magnetic current densities; finally, \( \varepsilon_{ijk} \) is the alternating tensor (or Levi–Civita tensor), with \( \varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1, \varepsilon_{213} = \varepsilon_{132} = \varepsilon_{123} = -1, \) and all other components being zero. The permittivity, permeability and conductivity obey the symmetry relations \( \varepsilon_{ik} = \varepsilon_{ik} \), \( \mu_{km} = \mu_{mk} \), and \( \sigma_{ik} = \sigma_{ki} \), respectively. Equations (A13) and (A14) can be combined into the general matrix-vector Eq. (1), with
\[ \mathbf{u} = \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}, \quad \mathbf{s} = \begin{pmatrix} -\mathbf{J}^e \\ -\mathbf{J}^m \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \varepsilon & \mathbf{O} \\ \mathbf{O} & \mathbf{\mu} \end{pmatrix}, \] \hspace{0.5cm} (A15)
\[ \mathbf{B} = \begin{pmatrix} \mathbf{\sigma} & \mathbf{O} \\ \mathbf{O} & \mathbf{\mu} \end{pmatrix}, \] \hspace{0.5cm} (A16)
\[ \mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix}, \quad \mathbf{J}^e = \begin{pmatrix} J^e_1 \\ J^e_2 \\ J^e_3 \end{pmatrix}, \] \hspace{0.5cm} (A17)
\[ \mathbf{J}^m = \begin{pmatrix} J^m_1 \\ J^m_2 \\ J^m_3 \end{pmatrix}, \] \hspace{0.5cm} (A18)
\[ \mathbf{D}_x = \begin{pmatrix} \mathbf{O} & \mathbf{D}_x^T \\ \mathbf{D}_0 & \mathbf{O} \end{pmatrix}, \quad \mathbf{D}_0 = \begin{pmatrix} 0 & -\partial_3 & -\partial_2 \\ -\partial_3 & 0 & -\partial_1 \\ -\partial_2 & -\partial_1 & 0 \end{pmatrix}, \] \hspace{0.5cm} (A19)
with \( \mathbf{O} \) being the \( 3 \times 3 \) null matrix. Note that \( \varepsilon = \varepsilon^T, \mathbf{\mu} = \mathbf{\mu}^T \) and \( \mathbf{\sigma} = \mathbf{\sigma}^T \). \( \mathbf{D}_x \) obeys symmetry relations (2) and (3), with \( \mathbf{K} \) defined as
\[ \mathbf{K} = \begin{pmatrix} -\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \end{pmatrix}, \] \hspace{0.5cm} (A20)
with \( \mathbf{I} \) being the \( 3 \times 3 \) identity matrix. The frequency-domain matrix \( \mathbf{A} \), defined in Eq. (6), is given by
\[ \mathbf{A}(x, \omega) = \begin{pmatrix} \varepsilon(x, \omega) & \mathbf{O} \\ \mathbf{O} & \mu(x) \end{pmatrix}, \] \hspace{0.5cm} (A21)
with
\[ \varepsilon(x, \omega) = \varepsilon(x) - \frac{\sigma(x)}{i\omega}. \] \hspace{0.5cm} (A22)
More generally, for bianisotropic materials this matrix becomes a full matrix, according to\(^{45,43,44} \)
\[ \mathbf{A}(x, \omega) = \begin{pmatrix} \varepsilon(x, \omega) & \xi(x, \omega) \\ \xi(x, \omega) & \mu(x, \omega) \end{pmatrix}. \] \hspace{0.5cm} (A23)
Note that
\[ A^{(a)} = K A^T K = \begin{pmatrix} e^T(x, \omega) & -\epsilon^T(x, \omega) \\ -\epsilon^T(x, \omega) & \mu^T(x, \omega) \end{pmatrix}. \] (A23)

When \( \zeta = -\epsilon^T \), we have \( A^{(a)} = A \), meaning that the medium is reciprocal.\(^{45}\) On the other hand, when \( \zeta = \epsilon^T \) the medium is non-reciprocal. Energy is conserved when \( A^T = A \). In all cases this requires \( \Im(\epsilon) = \Im(\mu) = 0 \). In addition, for reciprocal media it requires \( \Re(\epsilon) = \Re(\mu) = 0 \), which occurs in so-called chiral media.\(^{46}\) On the other hand, for nonreciprocal media obeying \( \zeta = \epsilon^T \), it requires \( \Im(\epsilon) = \Im(\xi) = 0 \), which occurs for example in so-called Faraday media.\(^{47}\)

4. Elastodynamic wave equation

The linearized equation of motion in a lossless solid reads\(^{38,48,49}\)
\[ \rho \partial_t v_1 - \partial_j \tau_{ij} = f_i, \] (A24)

where \( v_1 \) and \( \tau_{ij} \) are the particle velocity and stress tensor, respectively, associated to the elastodynamic wave field, \( \rho \) is the mass density of the medium and \( f_i \) the external volume force. The stress tensor is symmetric, i.e., \( \tau_{ij} = \tau_{ji} \). Hooke's linearized stress-strain relation reads
\[ -s_{ijkl}\partial_{k}\varepsilon_{li} + \left( \partial_{i}\varepsilon_{lj} + \partial_{j}\varepsilon_{li} \right)/2 = h_{ij}, \] (A25)

where \( h_{ij} \) is the external deformation rate, with \( h_{ij} = h_{ji} \), and \( s_{ijkl} \) is the compliance tensor, with \( s_{ijkl} = s_{ikjl} = s_{klij} = s_{klij} \).

Equations (A24) and (A25) can be combined to yield the general matrix-vector Eq. (1). To this end, rewrite these equations as
\[ \rho \partial_t v - D_1 \tau_1 - D_2 \tau_2 = f \] (A26)

and
\[ -s_{11}\partial_1\varepsilon_{11} - 2s_{12}\partial_1\varepsilon_{12} + D_1 v = h_1, \] (A27)
\[ -2s_{21}\partial_2\varepsilon_{12} - 4s_{22}\partial_2\varepsilon_{22} + D_2 v = h_2, \] (A28)

where
\[ v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{pmatrix}, \quad f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \quad h_1 = \begin{pmatrix} h_{11} \\ h_{22} \\ h_{33} \end{pmatrix}, \quad h_2 = \begin{pmatrix} 2h_{23} \\ 2h_{31} \\ 2h_{12} \end{pmatrix}. \] (A29)

5. Piezoelectric wave equation

The equations for coupled electromagnetic and elastodynamic waves in a lossless piezoelectric material reads\(^{43,50}\)
\[ \varepsilon_{ik}\partial_t E_k - \epsilon_{ijk}\partial_l H_k + d_{ijk}\partial_t \tau_{jk} = -J^E, \] (A37)
\[ \mu_{km}\partial_t H_m + \epsilon_{klm}\partial_t E_m = -J^H, \] (A38)

\[ \rho \partial_t v_i - \partial_j \tau_{ij} = f_i, \] (A39)

\[ -s_{ijkl}\partial_{k}\varepsilon_{li} + \left( \partial_{i}\varepsilon_{lj} + \partial_{j}\varepsilon_{li} \right)/2 - d_{ijk}\partial_t E_k = h_{ij}, \] (A40)

where \( d_{ijk} \) is the coupling tensor, with \( d_{ijk} = d_{ikj} = d_{kji} \). Note that \( \varepsilon_{ik} \) in Eq. (A37) and \( s_{ijkl} \) in Eq. (A40) are parameters measured under constant stress and constant electric field, respectively. Equations (A37)–(A40) can be combined into the general matrix-vector Eq. (1), with
\[ u = \begin{pmatrix} u_{EM} \\ u_{ED} \end{pmatrix}, \quad s = \begin{pmatrix} s_{EM} \\ s_{ED} \end{pmatrix}, \] (A41)
\[ A = \begin{pmatrix} A^{EM} & A^{C} \\ A^{C} & A^{ED} \end{pmatrix}, \quad D_x = \begin{pmatrix} D_{EM} & 0 \\ 0 & D_{ED} \end{pmatrix}. \] (A42)
and \( \mathbf{B} \) a 15 \( \times \) 15 null matrix. Superscripts EM and ED stand for electromagnetic and elastodynamic, respectively. The expressions for the wave field vectors, source vectors, medium parameter matrices, and differential operators with superscripts EM and ED are given in Appendices A3 and A4, respectively (but here only the lossless reciprocal case is considered). The coupling matrix \( \mathbf{A}^c \) is defined as follows:

\[
\mathbf{A}^c = \begin{pmatrix}
\mathbf{0} & -\mathbf{d}_1 & -2\mathbf{d}_2 \\
\mathbf{0} & \mathbf{0} & \mathbf{0}
\end{pmatrix},
\]

with

\[
\mathbf{d}_1 = \begin{pmatrix}
d_{11} & d_{12} & d_{13} \\
d_{21} & d_{22} & d_{23} \\
d_{31} & d_{32} & d_{33}
\end{pmatrix},
\quad \mathbf{d}_2 = \begin{pmatrix}
d_{11} & d_{123} & d_{112} \\
d_{21} & d_{231} & d_{212} \\
d_{31} & d_{331} & d_{312}
\end{pmatrix}.
\]

\( \mathbf{D}_x \) obeys symmetry relations (2) and (3), with \( \mathbf{K} \) defined as

\[
\mathbf{K} = \begin{pmatrix}
\mathbf{K}^{EM} & \mathbf{0} \\
\mathbf{0} & \mathbf{K}^{ED}
\end{pmatrix},
\]

with \( \mathbf{K}^{EM} \) and \( \mathbf{K}^{ED} \) defined in Appendixes A3 and A4, respectively. The frequency-domain matrix \( \mathbf{A} \), defined in Eq. (6), is identical to \( \mathbf{A}_s \), defined in Eq. (A42), because a lossless material is considered.

Note that \( \mathbf{A} = \mathbf{K} \mathbf{A}^T \mathbf{K} \), hence \( \mathbf{A}^{(a)} = \mathbf{A} \), meaning that reciprocity is obeyed. Furthermore, \( \mathbf{A}^\dagger = \mathbf{A} \), hence, energy is conserved.

**APPENDIX B: THEOREM OF GAUSS IN MATRIX-VECTOR FORM**

For a scalar field \( a(\mathbf{x}) \), the theorem of Gauss reads

\[
\int_{\partial \mathbf{D}} \partial_j a(\mathbf{x}) d^3 \mathbf{x} = \oint_{\partial \partial \mathbf{D}} a(\mathbf{x}) n_j d^2 \mathbf{x}. \tag{B1}
\]

Here this theorem is modified for the differential operator matrix \( \mathbf{D}_x \) appearing in Eqs. (1) and (5). Let \( \mathbf{D}_{IJ} \) denote the operator in row \( I \) and column \( J \) of matrix \( \mathbf{D}_x \). The symmetry of \( \mathbf{D}_x \) [Eq. (2)] implies \( \mathbf{D}_{IJ} = \mathbf{D}_{JI} \). Define a matrix \( \mathbf{N}_x \) which contains the components of the normal vector \( \mathbf{n} \) on \( \partial \partial \mathbf{D} \), organized in the same way as matrix \( \mathbf{D}_x \). Hence, \( \mathbf{N}_{IJ} = \mathbf{N}_{JI} \), where \( \mathbf{N}_{IJ} \) denotes the element in row \( I \) and column \( J \) of matrix \( \mathbf{N}_x \). Replace the scalar field \( a(\mathbf{x}) \) by \( a(\mathbf{x}) b_J(\mathbf{x}) \) and sum both sides of Eq. (B1) over \( I \) and \( J \). This yields

\[
\int_{\partial \mathbf{D}} \mathbf{D}_{IJ}(a(\mathbf{x}) b_J(\mathbf{x})) d^3 \mathbf{x} = \oint_{\partial \partial \mathbf{D}} a(\mathbf{x}) b_J(\mathbf{x}) n_J d^2 \mathbf{x}, \tag{B2}
\]

where the summation convention applies to repeated capital Latin subscripts, which may run from 1 to 4, 6, 9, or 15, depending on the choice of operator \( \mathbf{D}_x \). Applying the product rule for differentiation and using the symmetry property \( \mathbf{D}_{IJ} = \mathbf{D}_{JI} \) yields for the integrand in the left-hand side of Eq. (B2)

\[
\mathbf{D}_{IJ}(a b_J) = a(\mathbf{x}) \mathbf{D}_{IJ} b_J + (\mathbf{D}_{IJ} a(\mathbf{x})) b_J = a^T \mathbf{D}_x b + (\mathbf{D}_x a)^T b.
\]

For convenience \( G^{p,q} \) is renamed as \( G_0 \). All elements of matrix \( \mathbf{G}(\mathbf{x}, \mathbf{x'}) \) are now expressed in terms of \( G_0(\mathbf{x}, \mathbf{x'}) \), for \( \mathbf{x} \neq \mathbf{x}' \). Transforming the equation of motion (A1) to the frequency domain, gives for the first column of \( \mathbf{G}(\mathbf{x}, \mathbf{x'}) \),

\[
\left( \begin{array}{c}
\frac{1}{i\omega p} \\
\frac{\partial}{i\omega p} \\
\frac{1}{i\omega p}
\end{array} \right) G_0(\mathbf{x}, \mathbf{x'}), \tag{C3}
\]

\[\int_D \left\{ \mathbf{a}^T \mathbf{D}_x \mathbf{b} + (\mathbf{D}_x \mathbf{a})^T \mathbf{b} \right\} d^3 x = \int_{\partial D} a^T N_x b d^2 x. \tag{B4}\]

**APPENDIX C: GREEN’S MATRICES**

1. Acoustic Green’s matrix

The frequency-domain Green’s matrix \( \mathbf{G}(\mathbf{x}, \mathbf{x'}) \) is a \( L \times L \) matrix, obeying wave equation (15). The element in the \( k \)th row and \( l \)th column represents the wave field quantity of the \( k \)th type observed at \( \mathbf{x} \), due to a unit source of the \( l \)th type at \( \mathbf{x'} \). Here “wave field quantity of the \( k \)th type” means the wave field quantity represented by the \( k \)th element of wave field vector \( \mathbf{u} \). Similarly, “source of the \( l \)th type” means the type of source represented by the \( l \)th element of source vector \( \mathbf{s} \). Hence, for the acoustic situation the Green’s matrix can be written as

\[
\mathbf{G}(\mathbf{x}, \mathbf{x'}) = \begin{pmatrix}
G_{p,1}^{q,1} & G_{p,1}^{f,1} & G_{p,1}^{c,1} \\
G_{p,2}^{q,1} & G_{p,2}^{f,1} & G_{p,2}^{c,1} \\
G_{p,3}^{q,1} & G_{p,3}^{f,1} & G_{p,3}^{c,1}
\end{pmatrix}(\mathbf{x}, \mathbf{x'}). \tag{1.1}
\]

Superscripts \( p \) and \( q \) refer to the observations of acoustic pressure and particle velocity, respectively, at \( \mathbf{x} \), whereas superscripts \( f \) and \( c \) refer to sources of volume injection rate and external volume force, respectively, at \( \mathbf{x'} \). The subscripts refer to the components of the particle velocity and volume force, respectively.

A non-flowing acoustic medium is reciprocal, see Appendix A1. Hence, symmetry relation (18), with \( \mathbf{K} \) defined in Eq. (A5), gives

\[
\begin{pmatrix}
G_{p,1}^{q,1} & G_{p,1}^{f,1} & G_{p,1}^{c,1} \\
G_{p,2}^{q,1} & G_{p,2}^{f,1} & G_{p,2}^{c,1} \\
G_{p,3}^{q,1} & G_{p,3}^{f,1} & G_{p,3}^{c,1}
\end{pmatrix}
= \begin{pmatrix}
G_{q,1}^{p,1} & -G_{q,1}^{f,1} & -G_{q,1}^{c,1} \\
-G_{q,2}^{p,1} & G_{q,2}^{f,1} & G_{q,2}^{c,1} \\
-G_{q,3}^{p,1} & G_{q,3}^{f,1} & G_{q,3}^{c,1}
\end{pmatrix}, \tag{C2}
\]

where \( a \) and \( b \) are vector functions, containing the scalar functions \( a(\mathbf{x}) \) and \( b(\mathbf{x}) \), respectively. Rewriting the integrand in the right-hand side of Eq. (B2) in a similar way, gives the theorem of Gauss in matrix-vector form

\[
\int_D \left\{ \mathbf{a}^T \mathbf{D}_x \mathbf{b} + (\mathbf{D}_x \mathbf{a})^T \mathbf{b} \right\} d^3 x = \int_{\partial D} a^T N_x b d^2 x. \tag{B4}\]


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with \( \rho = \rho(x, \omega) \) defined in Eq. (A9). Similar expressions hold for the other columns of \( \mathbf{G}(x, x') \). Based on symmetry relation (C2), the first row of \( \mathbf{G}(x, x') \) can be expressed as

\[
\left( 1 - \frac{\partial f}{i\omega \partial x} - \frac{\partial g}{i\omega \partial x} - \frac{\partial h}{i\omega \partial x} \right) G_0(x, x'),
\]

where \( \partial f \) denotes differentiation with respect to \( x' \) and \( \rho' = \rho(x', \omega) \). Combining these two relations gives

\[
\mathbf{G}(x, x') = \left( \begin{array}{c} \frac{\partial}{i\omega} \\ \frac{\partial x}{i\omega} \\ \frac{\partial x}{i\omega} \end{array} \right) \left( 1 - \frac{\partial f}{i\omega \partial x} - \frac{\partial g}{i\omega \partial x} - \frac{\partial h}{i\omega \partial x} \right) G_0(x, x'),
\]

From here onward, this Green’s matrix is analyzed for a homogeneous lossless background medium. Replace \( G_0(x, x') \) by the background Green’s function

\[
\bar{G}_0(y) = \frac{1}{i\zeta} \exp(ik|y|),
\]

where

\[
\zeta = 4\pi/\omega \rho,
\]

with \( y = x - x' \) and \( k = \omega/c \), with propagation velocity \( c = (\kappa \rho)^{-1/2} \). Here \( \kappa \) and \( \rho \) are the compressibility and mass density of the background medium (for notational convenience, bars are omitted on the background medium parameters). In the far field approximation, Eq. (C5) gives

\[
\mathbf{G}(x, x') = \theta(y) \bar{G}_0(y) \theta^T(y),
\]

where

\[
\theta(y) = \left( \begin{array}{c} \hat{y}_1/pc \\ \hat{y}_2/pc \\ \hat{y}_3/pc \end{array} \right),
\]

with \( \hat{y}_i = y_i/|y| = (x_i - x'_i)/|x - x'| \).

Equations (C6) and (C8) are used to evaluate the term \( \mathbf{G}^+(x, 0) \mathbf{M}(x) \mathbf{G}(x, 0) \) in Eq. (36). Substitution of Eq. (C8) with \( x' = 0 \) gives

\[
\mathbf{G}^+(x, 0) \mathbf{M}(x) \mathbf{G}(x, 0) = \frac{\alpha^2 \rho^2}{16\pi^2 |x|^2} \theta(x) \theta^T(x) \mathbf{M}(x) \theta(x) \theta^T(x).
\]

Using \( \mathbf{M}(x) \) as defined in Eq. (A6), yields

\[
\theta^T(x) \mathbf{M}(x) \theta(x) = 2/pc.
\]

Hence

\[
\mathbf{G}^+(x, 0) \mathbf{M}(x) \mathbf{G}(x, 0) = \frac{2 \Theta(x)}{\zeta |x|^2},
\]

Next, Eqs. (C6) and (C8) are used to establish symmetry relation (46). Consider the Green’s function \( \mathbf{G}(x, x') \) as defined in Eq. (C8), with \( x \) in the scattering domain \( D_s \) and \( x' \) far from this scattering domain, hence, \( |x| \ll |x'| \), see Fig. 3. Express \( \mathbf{G}(x, x') \) as

\[
\mathbf{G}(x, x') = \mathbf{P}(x, x') \mathbf{G}(0, x'),
\]

where, according to Eq. (C8),

\[
\mathbf{G}(0, x') = \theta(-x') \mathbf{G}_0(-x') \theta^T(-x').
\]

An expression for \( \mathbf{P}(x, x') \) is derived by constructing \( \mathbf{G}(x, x') \), as defined in Eq. (C8), from \( \mathbf{G}(0, x') \), as defined in Eq. (C14), in three steps.

Step 1: using \( (1 0 0 0) \theta(-x') = 1 \), eliminate \( \theta(-x') \) from Eq. (C14) as follows

\[
(1 0 0 0) \mathbf{G}(0, x') = \mathbf{G}_0(-x') \theta^T(-x').
\]

Step 2: using \( \mathbf{G}_0(y) \approx \exp(-ikx \cdot y) \mathbf{G}_0(x') \), applying \( \exp(-ikx \cdot y) \) to the right-hand side of Eq. (C15) gives

\[
\exp(-ikx \cdot y) \mathbf{G}_0(-x') \theta^T(-x') \approx \mathbf{G}_0(y) \theta^T(-x').
\]

Step 3: \( \mathbf{G}(x, x') \) is obtained by applying \( \theta(-x') \) to the right-hand side of Eq. (C16) and using \( -x' \approx y \) and Eq. (C8). Hence

\[
\theta(-x') \mathbf{G}_0(y) \theta^T(-x') \approx \mathbf{G}(x, x').
\]

Combining these three steps, yields

\[
\tilde{\mathbf{G}}(x, x') = \theta(-x') \exp(-ikx \cdot y) \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right) \mathbf{G}(0, x').
\]

The quantum-mechanical Green’s matrix is similar to the acoustic Green’s matrix. \( \mathbf{G}_0(y) \) is again given by Eq. (C6), with

\[
\Theta(x) = \frac{k}{4\pi} \theta(x) \theta^T(x).
\]

Note that

\[
\mathbf{P}(x, x') = \mathbf{K} \mathbf{P}^*(x, -x') \mathbf{K},
\]

with \( \mathbf{K} \) defined in Eq. (A5), which confirms Eq. (46).
\( \zeta = 4\pi \hbar \) \hspace{1cm} (C21)

and \( k = \sqrt{2\omega m/\hbar} \). The far-field approximation of the Green’s matrix \( G(\mathbf{x}, \mathbf{x}') \) is again given by Eq. (C8), with

\[
\begin{bmatrix}
1 \\
\hbar k \hat{y}_1 \\
\hbar k \hat{y}_2 \\
\hbar k \hat{y}_3
\end{bmatrix}.
\] (C22)

Despite these different definitions, we find that the term \( G^\dagger(\mathbf{x}, 0)\mathbf{M}(\mathbf{x})\mathbf{G}(\mathbf{x}, 0) \) in Eq. (36) can be expressed again by Eqs. (C11) and (C12). Moreover, matrix \( \mathbf{P}(\mathbf{x}, \mathbf{x'}) \) obeys again symmetry relation (C20).

3. Electromagnetic Green’s matrix

The basic \( 3 \times 3 \) far-field electromagnetic Green’s matrix in a homogeneous, isotropic, reciprocal, lossless background is given by38

\[
\mathbf{G}_0(\mathbf{y}) = \frac{\mu \exp(\nu k |\mathbf{y}|)}{i\zeta} \{ \Gamma(\mathbf{\hat{y}}) - I\},
\] (C23)

where

\[
\Gamma(\mathbf{\hat{y}}) = \begin{pmatrix}
\hat{y}_1^2 & \hat{y}_1 \hat{y}_2 & \hat{y}_1 \hat{y}_3 \\
\hat{y}_2 \hat{y}_1 & \hat{y}_2^2 & \hat{y}_2 \hat{y}_3 \\
\hat{y}_3 \hat{y}_1 & \hat{y}_3 \hat{y}_2 & \hat{y}_3^2
\end{pmatrix},
\] (C25)

and \( k = \omega/c \), with propagation velocity \( c = (\varepsilon_0 \mu)^{-1/2} \). Here \( \varepsilon \) and \( \mu \) are the permittivity and permeability of the background. Analogous to the derivation in Appendix C1 it can be shown that the \( 6 \times 6 \) Green’s matrix \( \mathbf{G}(\mathbf{x}, \mathbf{x'}) \) is, in the far field, related to the basic \( 3 \times 3 \) matrix \( \mathbf{G}_0(\mathbf{y}) \), via

\[
\mathbf{G}(\mathbf{x}, \mathbf{x'}) = \mathbf{\theta}(\mathbf{\hat{y}}) \mathbf{G}_0(\mathbf{y}) \mathbf{\theta}^T(\mathbf{\hat{y}}),
\] (C26)

with

\[
\mathbf{\theta}(\mathbf{\hat{y}}) = \begin{pmatrix}
\mathbf{I} \\
\frac{1}{\mu \nu} \mathbf{M}_0(\mathbf{\hat{y}})
\end{pmatrix}, \quad
\mathbf{M}_0(\mathbf{\hat{y}}) = \begin{pmatrix}
0 & -\hat{y}_3 & -\hat{y}_2 \\
\hat{y}_3 & 0 & -\hat{y}_1 \\
\hat{y}_2 & \hat{y}_1 & 0
\end{pmatrix}.
\] (C27)

Equations (C23) and (C26) are used to evaluate the term \( \mathbf{G}^\dagger(\mathbf{x}, 0)\mathbf{M}(\mathbf{x})\mathbf{G}(\mathbf{x}, 0) \) in Eq. (36). Substitution of Eq. (C26) with \( \mathbf{x} = 0 \) gives

\[
\mathbf{G}^\dagger(\mathbf{x}, 0)\mathbf{M}(\mathbf{x})\mathbf{G}(\mathbf{x}, 0) = \mathbf{\theta}(\mathbf{\hat{x}}) \left\{ \mathbf{G}_0(\mathbf{0}) \right\} \mathbf{\theta}^T(\mathbf{\hat{x}}) \mathbf{G}_0(\mathbf{x}) \mathbf{\theta}^T(\mathbf{\hat{x}}) \cdot
\] (C28)

Analogous to the definition of \( \mathbf{D}_\mathbf{x} \) in Eq. (A18), it holds that

\[
\mathbf{M}(\mathbf{x}) = \begin{pmatrix}
\mathbf{0} & \mathbf{M}_0(\mathbf{x}) \\
\mathbf{M}_0^T(\mathbf{x}) & \mathbf{0}
\end{pmatrix},
\] (C29)

hence

\[
\mathbf{\theta}^T(\mathbf{x})\mathbf{M}(\mathbf{x})\mathbf{\theta}(\mathbf{\hat{x}}) = \frac{2}{\mu c} \mathbf{M}_0^T(\mathbf{x})\mathbf{M}_0(\mathbf{x})
\]

\[= \frac{2}{\mu c} \{ I - \Gamma(\mathbf{x}) \}. \] (C30)

Substituting this into Eq. (C28), using

\[\Gamma = \Gamma^T = \Gamma^2 = \Gamma^3 = \cdots, \] (C31)

gives

\[
\mathbf{G}^\dagger(\mathbf{x}, 0)\mathbf{M}(\mathbf{x})\mathbf{G}(\mathbf{x}, 0) = \frac{2}{\mu c} \frac{\Theta(\mathbf{x})}{\zeta |\mathbf{x}|^2},
\] (C32)

with

\[
\Theta(\mathbf{x}) = \frac{\mu k}{4\pi} \mathbf{\theta}(\mathbf{\hat{x}}) \{ I - \Gamma(\mathbf{x}) \} \mathbf{\theta}^T(\mathbf{\hat{x}}). \] (C33)

Next, the same three steps as in Appendix C1 are applied to establish symmetry relation (46). Assuming \( |\mathbf{x}| \ll |\mathbf{x}'| \) (Fig. 3), express \( \mathbf{G}(\mathbf{x}, \mathbf{x'}) \) as

\[
\mathbf{G}(\mathbf{x}, \mathbf{x'}) = \mathbf{P}(\mathbf{x}, \mathbf{x'}) \mathbf{G}(\mathbf{0}, \mathbf{x'}),
\] (C34)

where, analogous to Eq. (C19),

\[
\mathbf{P}(\mathbf{x}, \mathbf{x'}) = \mathbf{\theta}(-\mathbf{\hat{x}}') \exp(-ik\mathbf{x'} \cdot \mathbf{\hat{x}}') \{ I \ O \}
\]

\[= \begin{pmatrix}
I \\
\frac{1}{\mu c} \mathbf{M}_0(-\mathbf{x'}) \\
O
\end{pmatrix} \exp(-ik\mathbf{x'} \cdot \mathbf{\hat{x}}').\] (C35)

Note that

\[
\mathbf{P}(\mathbf{x}, \mathbf{x'}) = \mathbf{K} \mathbf{P}^*(\mathbf{x}, -\mathbf{x'}) \mathbf{K},
\] (C36)

with \( \mathbf{K} \) defined in Eq. (A19), which confirms Eq. (46).

4. Elastodynamic Green’s matrix

The basic \( 3 \times 3 \) far-field elastodynamic Green’s matrices for \( P \) - and \( S \) -waves in a homogeneous, isotropic, lossless background medium are given by69

\[
\mathbf{G}_P(\mathbf{y}) = \frac{1}{i\zeta \rho c_P^2} \frac{\exp(ikP|\mathbf{y}|)}{|\mathbf{y}|} \Gamma(\mathbf{\hat{y}})
\] (C37)

and

\[
\mathbf{G}_S(\mathbf{y}) = \frac{1}{i\zeta \rho c_S^2} \frac{\exp(ikS|\mathbf{y}|)}{|\mathbf{y}|} \{ I - \Gamma(\mathbf{\hat{y}}) \},
\] (C38)

respectively, where

\[\zeta = 4\pi/\omega, \] (C39)

\[\Gamma(\mathbf{\hat{y}}) \] defined by Eq. (C25), and \( k_{\{P, S\}} = \omega/c_{\{P, S\}} \), with propagation velocities \( c_P = \sqrt{(\lambda + 2\mu)/\rho} \) and \( c_S = \sqrt{\mu/\rho} \). Here \( \lambda, \mu, \) and \( \rho \) are the Lamé parameters and mass density of the background medium. Analogous to the derivation in Appendix C1 it can be shown that the \( 9 \times 9 \) Green’s matrix \( \mathbf{G}(\mathbf{x}, \mathbf{x'}) \) is, in the far field, related to the basic \( 3 \times 3 \) matrices \( \mathbf{G}_P(\mathbf{y}) \) and \( \mathbf{G}_S(\mathbf{y}) \), via
\[
\bar{G}(x, x') = \theta_p(\hat{y})G_p(y)\theta_p^T(\hat{y}) + \theta_s(\hat{y})G_s(y)\theta_s^T(\hat{y}),
\]
where
\[
\theta_{(P,S)}(\hat{y}) = \begin{pmatrix}
\frac{1}{c_{(P,S)}} c_{11} M_1(\hat{y}) \\
\frac{1}{c_{(P,S)}} c_{22} M_2(\hat{y})
\end{pmatrix},
\]
with
\[
c_{11} = \begin{pmatrix}
\lambda + 2\mu & \lambda & \lambda \\
\lambda & \lambda + 2\mu & \lambda \\
\lambda & \lambda & \lambda + 2\mu
\end{pmatrix},
\]
\[
c_{22} = \begin{pmatrix}
\mu & 0 & 0 \\
0 & \mu & 0 \\
0 & 0 & \mu
\end{pmatrix},
\]
and
\[
M_1(\hat{y}) = \begin{pmatrix}
\hat{y}_1 & 0 & 0 \\
0 & \hat{y}_2 & 0 \\
0 & 0 & \hat{y}_3
\end{pmatrix},
\]
\[
M_2(\hat{y}) = \begin{pmatrix}
0 & \hat{y}_3 & \hat{y}_2 \\
\hat{y}_3 & 0 & \hat{y}_1 \\
\hat{y}_2 & \hat{y}_1 & 0
\end{pmatrix}.
\]

We use Eqs. (C37), (C38), and (C40) to evaluate the term \(G^T(x,0)M(\hat{x})G(x,0)\) in Eq. (36). Substitution of Eq. (C40) with \(x' = 0\) gives
\[
G^T(x,0)M(\hat{x})G(x,0) = \{\theta_p(\hat{x})G_p(x)\theta_p^T(\hat{x}) + \theta_s(\hat{x})G_s(x)\theta_s^T(\hat{x})\} \{\theta_p(\hat{x})G_p(x)\theta_p^T(\hat{x}) + \theta_s(\hat{x})G_s(x)\theta_s^T(\hat{x})\}.
\]

Analogous to the definition of \(D_\lambda\) in Eq. (A35) it holds that
\[
M(\hat{x}) = \begin{pmatrix}
O & M_1(\hat{x}) & M_2(\hat{x}) \\
M_1(\hat{x}) & O & O \\
M_2(\hat{x}) & O & O
\end{pmatrix}.
\]

Consider the terms \(\theta_Q^T(\hat{x})M(\hat{x})\theta_R(\hat{x})\), where each of the subscripts \(Q\) and \(R\) can stand for either \(P\) or \(S\). Using Eqs. (C41)–(C43) gives
\[
\theta_Q^T(\hat{x})M(\hat{x})\theta_R(\hat{x}) = \left(\frac{1}{c_Q} + \frac{1}{c_R}\right) (M_1(\hat{x})c_{11} M_1(\hat{x}) + M_2(\hat{x})c_{22} M_2(\hat{x})) = \left(\frac{1}{c_Q} + \frac{1}{c_R}\right) ((\lambda + \mu)\Gamma(\hat{x}) + \mu I),
\]
with \(\Gamma(\hat{x})\) defined by Eq. (C25). Hence, using Eq. (C31), it is found for the different terms in Eq. (C44) that
\[
G_p^T(x)\theta_p^T(\hat{x})M(\hat{x})\theta_p(\hat{x})G_p(x) = \frac{\omega^2 \Gamma(\hat{x})}{8\rho c_p^3 \pi^2 |x|^2},
\]
\[
G_s^T(x)\theta_s^T(\hat{x})M(\hat{x})\theta_s(\hat{x})G_s(x) = \frac{\omega^2 (1 - \Gamma(\hat{x}))}{8\rho c_s^3 \pi^2 |x|^2}.
\]
Taking all terms together yields
\[
G^T(x,0)M(x)G(x,0) = \frac{2\Theta(\hat{x})}{\zeta |x|^2},
\]
with
\[
\Theta(\hat{x}) = \frac{\omega}{4\pi\rho} \left(\frac{1}{c_p} \theta_p(\hat{x})\Gamma(\hat{x})\theta_p^T(\hat{x}) + \frac{1}{c_s} \theta_s(\hat{x}) (1 - \Gamma(\hat{x})) \theta_s^T(\hat{x})\right).
\]

Next, Eqs. (C37), (C38), and (C40) are used to establish symmetry relation (46). Assuming \(|x| \ll |x'|\) (Fig. 3), express \(\bar{G}(x,x')\) as
\[
\bar{G}(x,x') = \bar{P}(x,x')\bar{G}(0, x'),
\]
where, according to Eq. (C40),
\[
\bar{G}(0, x') = \theta_p(-x')\bar{G}_p(-x')\theta_p^T(-x') + \theta_s(-x')\bar{G}_s(-x')\theta_s^T(-x').
\]

An expression for \(\bar{P}(x,x')\) is derived by constructing \(\bar{G}(x,x')\), as defined in Eq. (C40), from \(\bar{G}(0, x')\), as defined in Eq. (C54), in three steps.

Step 1: using \(\Gamma(x') = \Gamma(-x')\) as well as Eq. (C31), decompose \(\bar{G}(0, x')\) into its \(P\)- and \(S\)-wave constituents, as follows:
\[
\begin{pmatrix}
\Gamma(\hat{x}) & O & O \\
O & I - \Gamma(\hat{x}) & O \\
O & O & I - \Gamma(\hat{x})
\end{pmatrix} \bar{G}(0, x') = \begin{pmatrix}
\bar{G}_p(-x')\theta_p^T(-x') \\
\bar{G}_s(-x')\theta_s^T(-x')
\end{pmatrix}.
\]

Step 2: using \(\bar{G}_{(P,S)}(y) \approx \exp(-ik_{(P,S)}x \cdot \hat{x}')\bar{G}_{(P,S)}(-x')\), applying \(\exp(-ik_{(P,S)}x \cdot \hat{x}')\) to the right-hand side of Eq. (C55) gives
\[
\begin{pmatrix}
I \exp(-ik_{(P,S)}x \cdot \hat{x}') & O \\
O & I \exp(-ik_{(P,S)}x \cdot \hat{x}')
\end{pmatrix} \begin{pmatrix}
\bar{G}_p(-x')\theta_p^T(-x') \\
\bar{G}_s(-x')\theta_s^T(-x')
\end{pmatrix} = \begin{pmatrix}
\bar{G}_p(y)\theta_p^T(-x') \\
\bar{G}_s(y)\theta_s^T(-x')
\end{pmatrix},
\]

Step 3: compose \(\bar{G}(x,x')\) from its \(P\)- and \(S\)-wave constituents by applying \(\theta_p(-x')\theta_s(-x')\) to the right-hand side of Eq. (C56) and using \(-x' \approx \hat{y}\) and Eq. (C40). Hence
\[
\bar{G}(x,x') = \theta_p(-x')\theta_s(-x') \begin{pmatrix}
\bar{G}_p(y)\theta_p^T(-x') \\
\bar{G}_s(y)\theta_s^T(-x')
\end{pmatrix} = \bar{G}(x,x').
\]

Combining these three steps gives Eq. (C53), with
\[ \hat{P}(\mathbf{x}, \mathbf{x}') = (\hat{P}_1(\mathbf{x}, \mathbf{x}') \quad \mathbf{O} \quad \mathbf{O}), \]  

where

\[ \hat{P}_1(\mathbf{x}, \mathbf{x}') = \theta_p(\mathbf{x}', \mathbf{x}) \Gamma(\mathbf{x}') \exp(-ik_p \mathbf{x} \cdot \mathbf{x}') + \theta_c(\mathbf{x}', \mathbf{x}) \{I - \Gamma(\mathbf{x}')\} \exp(-ik_c \mathbf{x} \cdot \mathbf{x}'), \]  

or

\[ \hat{P}(\mathbf{x}, \mathbf{x}') = \begin{pmatrix} \hat{P}_{11}(\mathbf{x}, \mathbf{x}') & \mathbf{O} & \mathbf{O} \\ \hat{P}_{21}(\mathbf{x}, \mathbf{x}') & \mathbf{O} & \mathbf{O} \\ \hat{P}_{31}(\mathbf{x}, \mathbf{x}') & \mathbf{O} & \mathbf{O} \end{pmatrix}, \]  

where

\[ \hat{P}_{11}(\mathbf{x}, \mathbf{x}') = \Gamma(\mathbf{x}') \exp(-ik_p \mathbf{x} \cdot \mathbf{x}') + \{I - \Gamma(\mathbf{x}')\} \exp(-ik_c \mathbf{x} \cdot \mathbf{x}'), \]  

\[ \hat{P}_{21}(\mathbf{x}, \mathbf{x}') = c_{11} M_{1}(\mathbf{x}') \Gamma(\mathbf{x}') \exp(-ik_p \mathbf{x} \cdot \mathbf{x}') + c_{5} \{I - \Gamma(\mathbf{x}')\} \exp(-ik_c \mathbf{x} \cdot \mathbf{x}'), \]  

\[ \hat{P}_{31}(\mathbf{x}, \mathbf{x}') = c_{23} M_{2}(\mathbf{x}') \Gamma(\mathbf{x}') \exp(-ik_p \mathbf{x} \cdot \mathbf{x}') + c_{5} \{I - \Gamma(\mathbf{x}')\} \exp(-ik_c \mathbf{x} \cdot \mathbf{x}'). \]  

Next, express \( \hat{G}(\mathbf{x}, \mathbf{x}') \) as

\[ \hat{G}(\mathbf{x}, \mathbf{x}') = \hat{P}(\mathbf{x}, \mathbf{x}') \hat{G}(0, \mathbf{x}'), \]  

where

\[ \hat{P}(\mathbf{x}, \mathbf{x}') = \begin{pmatrix} \hat{P}_{EM}(\mathbf{x}, \mathbf{x}') & \mathbf{O} & \mathbf{O} \\ \hat{P}_{ED}(\mathbf{x}, \mathbf{x}') & \mathbf{O} & \mathbf{O} \end{pmatrix}. \]  

Note that

\[ \hat{P}(\mathbf{x}, \mathbf{x}') = \mathbf{K} \hat{P}^{*}(\mathbf{x}, -\mathbf{x}') \mathbf{K}, \]  

with \( \mathbf{K} \) defined in Eq. (A45), which confirms Eq. (46).

5. Combined electromagnetic and elastodynamic Green’s matrix

For a homogeneous, isotropic, lossless background medium, in which electromagnetic and elastodynamic waves propagate independently, the Green’s matrices can be combined as follows:

\[ \hat{G}(\mathbf{x}, \mathbf{x}') = \begin{pmatrix} \hat{G}_{EM}(\mathbf{x}, \mathbf{x}') & \mathbf{O} & \mathbf{O} \\ \hat{G}_{ED}(\mathbf{x}, \mathbf{x}') & \mathbf{O} & \mathbf{O} \end{pmatrix}, \]  

where superscripts EM and ED stand for electromagnetic and elastodynamic, respectively. The expressions for matrices with superscripts EM and ED are given in Appendixes C3 and C4, respectively. For the term \( \hat{G}(\mathbf{x}, 0) \hat{M}(\mathbf{x}) \hat{G}(\mathbf{x}, 0) \) appearing in Eq. (36), with

\[ \hat{M}(\mathbf{x}) = \begin{pmatrix} M_{EM}(\mathbf{x}) & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & M_{ED}(\mathbf{x}) \end{pmatrix}, \]  

it is found that

\[ \hat{G}(\mathbf{x}, 0) \hat{M}(\mathbf{x}) \hat{G}(\mathbf{x}, 0) = \frac{2 \Theta(\mathbf{x})}{k} \frac{\zeta}{|\mathbf{x}|^2}, \]  

where \( \zeta = 4\pi/\zeta_0 \), and

\[ \Theta(\mathbf{x}) = \begin{pmatrix} \Theta_{EM}(\mathbf{x}) & \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \Theta_{ED}(\mathbf{x}) \end{pmatrix}. \]  


