A single-sided representation for the homogeneous Green’s function of a unified scalar wave equation

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A unified scalar wave equation is formulated, which covers 3D acoustic waves, 2D horizontally-polarised shear waves, 2D transverse electric EM waves, 2D transverse magnetic EM waves, 3D quantum-mechanical waves and 2D flexural waves. The homogeneous Green’s function of this wave equation is a combination of the causal Green’s function and its time-reversal, such that their singularities at the source position cancel each other. A classical representation expresses this homogeneous Green’s function as a closed boundary integral. This representation finds applications in holographic imaging, time-reversed wave propagation and Green’s function retrieval by cross correlation. The main drawback of the classical representation in those applications is that it requires access to a closed boundary around the medium of interest, whereas in many practical situations the medium can be accessed from one side only. Therefore a single-sided representation is derived for the homogeneous Green’s function of the unified scalar wave equation. Like the classical representation, this single-sided representation fully accounts for multiple scattering. The single-sided representation has the same applications as the classical representation, but unlike the classical representation it is applicable in situations where the medium of interest is accessible from one side only.

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I. INTRODUCTION

The homogeneous Green’s function is defined as the combination of the causal Green’s function and its time-reversed function. It can be represented in an exact form by a closed boundary integral [1]. This representation finds its applications in optical, acoustic and seismic holographic imaging methods [1–5]. Moreover, it plays an important role in the fields of time-reversed wave propagation [6–8] and Green’s function retrieval from controlled-source or ambient-noise data by cross correlation [9–12]. In many situations the closed boundary integral is for practical reasons replaced by an open boundary integral, for example because the medium to be imaged can be accessed from one side only. This induces approximations in the representation of the homogeneous Green’s function, which can be quite severe when there is significant multiple scattering due to inhomogeneities in the medium. In recent work we have derived a single-sided homogeneous Green’s function representation, which circumvents the need of omni-directional accessibility of the medium but nevertheless accounts for multiple scattering [13]. Subsequently we derived a unified form of this representation for a specific class of vectorial fields in lossless media [14]. Here we derive a unified form of the single-sided homogeneous Green’s function representation for a broad class of scalar fields in media with losses.

The setup of this paper is as follows. We start by introducing a unified scalar wave equation [15] and discuss briefly how the coefficients and operators in this wave equation are defined for a variety of scalar wave phenomena. Next we derive general reciprocity theorems (of the convolution type and of the correlation type) for two independent wave fields obeying the unified wave equation. We use the unified reciprocity theorem of the correlation type to derive a closed-boundary homogeneous Green’s function representation and show that the classical representation [1] follows as a special case of this. Next we modify the reciprocity theorems and derive a unified form of the single-sided homogeneous Green’s function representation. We briefly discuss some applications and end with some conclusions.

II. UNIFIED SCALAR WAVE EQUATION

Consider the unified scalar wave equation [15]

$$\mathcal{D} = \sum_{n=0}^{N} a_n \frac{\partial^n}{\partial t^n} u(x, t) = -\frac{\partial s(x, t)}{\partial t}. \quad (1)$$

Here $u(x, t)$ represents the scalar wave field as a function of space $(x = (x_1, x_2, x_3))$ and time $(t)$. Throughout this paper, the positive $x_3$-axis is pointing downward. The source distribution, which causes the wave field, is denoted as $s(x, t)$. The operator on the left-hand side consists of temporal differential operators, multiplied by space-dependent isotropic coefficients $a_n$, and an operator $\mathcal{D}$, containing spatial differential operators and space-dependent isotropic coefficients.

The temporal Fourier transform of a space- and time-dependent function $u(x, t)$ is defined as

$$u(x, \omega) = \int_{-\infty}^{\infty} \exp(i\omega t) u(x, t) dt, \quad (2)$$

where $\omega$ is the angular frequency and $i$ the imaginary unit. Note that, for notational convenience, the same
symbol \((u)\) is used for time- and frequency-domain functions. Equation (1) transformed to the frequency domain reads
\[
W u(x, \omega) = i \omega s(x, \omega),
\]
with the wave operator \(W\) defined as
\[
W = D - \sum_{n=0}^{N} (-i \omega)^n a_n.
\]
In this paper we consider two specific forms of the operator \(D\), named \(D_2\) and \(D_4\), but any other operator obeying the integral property discussed in section III (equation 9) may be chosen as well. For most wave phenomena considered in this paper, operator \(D\) is defined as
\[
D_2 = \partial_i b \partial_i,
\]
with \(\partial_i\) standing for spatial differentiation in the \(x_i\)-direction. Einstein’s summation convention applies to repeated subscripts. The subscript 2 in \(D_2\) denotes this is a second order differential operator. The notation in the right-hand side of equation (5) should be understood in the sense that differential operators act on everything to the right of it, hence, \(D_2 f\) stands for \(\partial_i (b \partial_i f)\).

Table 1: Definition of the wave fields, sources, operator and coefficients in wave equation (3) for the wave phenomena considered in this paper.

<table>
<thead>
<tr>
<th>Wave Type</th>
<th>Field</th>
<th>Source 1</th>
<th>Source 2</th>
<th>Operator</th>
<th>Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Acoustic waves (3D)</td>
<td>(u)</td>
<td>(s)</td>
<td>(a_0)</td>
<td>(a_1)</td>
<td>(a_2)</td>
</tr>
<tr>
<td>SH-waves (2D)</td>
<td>(v_2)</td>
<td>(f_2)</td>
<td>(r)</td>
<td>(\rho)</td>
<td>(\mu)</td>
</tr>
<tr>
<td>TE-waves (2D)</td>
<td>(E_2)</td>
<td>(-J_2^r)</td>
<td>(\sigma)</td>
<td>(\epsilon)</td>
<td>(\frac{1}{\rho})</td>
</tr>
<tr>
<td>TM-waves (2D)</td>
<td>(H_2)</td>
<td>(-J_2^m)</td>
<td>0</td>
<td>(\mu)</td>
<td>(\frac{1}{\rho \mu})</td>
</tr>
<tr>
<td>Quantum waves (3D)</td>
<td>(\psi)</td>
<td>(V)</td>
<td>(-i\hbar)</td>
<td>(\frac{\hbar^2}{2m})</td>
<td></td>
</tr>
<tr>
<td>Flexural waves (2D)</td>
<td>(v_2)</td>
<td>(f_2)</td>
<td>0</td>
<td>(\rho)</td>
<td>(\frac{2\omega}{\sqrt{pd}})</td>
</tr>
</tbody>
</table>

Table 1 shows the definitions of \(u, s, a_n, D\) and \(b\) for the different wave phenomena considered in this paper (with in all cases \(N = 2\)). The quantities in the first row correspond to the 3D acoustic wave equation in a fluid, with \(p\) the acoustic pressure, \(q\) the volume density of volume injection rate, \(\eta\) the viscosity, \(\kappa\) the compressibility and \(\rho\) the volume density of mass [16]. The second row refers to the 2D elastic wave equation for horizontally-polarised shear waves (SH) in a solid, with \(v_2\) the transverse particle velocity, \(f_2\) the volume density of transverse external force, \(r\) the coefficient of frictional force, \(\rho\) the volume density of mass and \(\mu\) the shear modulus [17]. In this paper, 2D wave equations are defined in the \(x = (x_1, x_3)\)-plane, hence, for the 2D situations, subscripts \(i\) in equation (5) takes on the values 1 and 3 only. The third and fourth rows refer to 2D electromagnetic wave equations in matter, with TE and TM standing for “transverse electric” and “transverse magnetic”, respectively. Here \(E_2\) and \(H_2\) are the transverse electric and transverse magnetic field strengths, respectively, \(J_2^r\) and \(J_2^m\) the volume densities of transverse external electric and magnetic current, respectively, \(\sigma\) the conductivity, \(\epsilon\) the permittivity and \(\mu\) the permeability [17, 18]. The fifth row corresponds to the 3D quantum wave equation (Schrödinger equation) for a particle with mass \(m\) in a potential \(V\), with \(\psi\) the quantum mechanical wave function, and \(\hbar = h/2\pi\), with \(h\) Planck’s constant [19, 20].

For flexural waves in a thin plate in the \(x = (x_1, x_3)\)-plane, operator \(D\) is defined as
\[
D_4 = -\partial_i \partial_i \partial_j \partial_j - \partial_i \partial_i d_i \partial_j \partial_j,
\]
with
\[
d_1 = (1 - \nu) d, \quad \text{(7)}
\]
\[
d_2 = \nu d, \quad \text{(8)}
\]
where \(\nu\) is Poisson’s ratio and \(d\) the bending stiffness (Appendix A). The subscript 4 in \(D_4\) denotes this is a fourth order differential operator. Subscripts \(i\) and \(j\) take on the values 1 and 3 only. For flexural waves, the quantities in the sixth row in Table 1 are the transverse particle velocity \(v_2\), the area density of transverse external force \(f_2\), and the area density of mass \(\rho\). The expression given for coefficient \(b\) in this row can for the moment be ignored. It will be discussed later.

To account for general loss mechanisms, we assume that \(a_n\) as well as the coefficients in operator \(D\) may be complex-valued and frequency-dependent, i.e., \(a_n = a_n(x, \omega)\), \(b = b(x, \omega)\), \(d_1 = d_1(x, \omega)\) and \(d_2 = d_2(x, \omega)\), with (for positive \(\omega\)) \(\Re\{(-i)^n a_n, b, d_1, d_2\} \geq 0\) and \(\Im\{(-i)^n a_n, b, d_1, d_2\} \leq 0\), where \(\Re\) and \(\Im\) denote the real and imaginary parts, respectively.

### III. RECIPROCITY THEOREMS

#### A. Integral property of operators \(D\) and \(W\)

For the derivation of the unified reciprocity theorems in the next two subsections we will make use of an integral property of the operators \(D\) and \(W\). Consider an arbitrary spatial domain \(V\), enclosed by boundary \(S\), with outward pointing normal vector \(n\) (Figure 1). For the 3D situation, the domain is a 3D volume and its boundary is a 2D surface with normal vector \(n = (n_1, n_2, n_3)\). For
the 2D situation, the domain is a 2D area in the \((x_1, x_3)\)-plane and its boundary is a 1D contour with normal vector \(n = (n_1, n_3)\).

Using the theorem of Gauss, it can be shown that

\[
\int_V [f(Dg) - (Df)g] \, dx = \oint_S J(f, g) \, ds,
\]

(9)

where, for \(D = D_2\), the interaction quantity \(J\) is defined as

\[
J_2(f, g) = [f(b \partial_t g) - (b \partial_t f)g] |n|,
\]

(10)

and, for \(D = D_4\),

\[
J_4(f, g) = \left[ (\partial_j f)(d_j \partial_t g) - (d_j \partial_t f)(\partial_j g) \right.
+ (\partial_j g)(d_j \partial_t f) - (d_j \partial_t g)(\partial_j f)
\]

\[
+ (\partial_j \partial_t f)(d_j \partial_t g) - (d_j \partial_t \partial_t f)(\partial_j g)
\]

\[
\left. + (\partial_j \partial_t g)(d_j \partial_t f) - (d_j \partial_t \partial_t g)(\partial_j f) \right] |n|.
\]

(11)

Here \(f = f(x)\) and \(g = g(x)\) are arbitrary space-dependent functions (not necessarily solutions of equation (3)). The left-hand side of equation (9) is unaltered when we add an arbitrary function to \(D\). In particular, we may replace \(D\) by \(W\), as defined in equation (4), hence

\[
\int_V [f(Wg) - (Wf)g] \, dx = \oint_S J(f, g) \, ds.
\]

(12)

\section*{B. Reciprocity theorem of the convolution type}

We consider two independent wave states \(A\) and \(B\) for which we derive a reciprocity theorem. The source distributions and wave fields in these states are distinguished by subscripts \(A\) and \(B\). Outside a sphere (or, in 2D, a circle) with finite radius, the medium (or, for quantum mechanics, the potential) is homogeneous and lossless in both states (Figure 1).

For the derivation of the first reciprocity theorem, the coefficients \(a_n, b_n\), etc. in \(V\) are chosen the same in both states (outside \(V\) the coefficients in the two states may be different). Hence, in the frequency domain the wave fields \(u_A(x, \omega)\) and \(u_B(x, \omega)\) obey in \(V\) the following two equations

\[
Wu_A(x, \omega) = i\omega s_A(x, \omega),
\]

(13)

\[
Wu_B(x, \omega) = i\omega s_B(x, \omega).
\]

(14)

We obtain a reciprocity theorem by substituting \(f = u_A\) and \(g = u_B\) into equation (12), using equations (13) and (14). We thus obtain

\[
i\omega \int_V [u_A s_B - s_A u_B] \, dx = \oint_S J(u_A, u_B) \, ds.
\]

(15)

We call this the reciprocity theorem of the convolution type [17, 21], because products like \(u_A s_B\) in the frequency domain correspond to convolutions in the time domain.

\section*{C. Reciprocity theorem of the correlation type}

We consider again two independent wave states \(A\) and \(B\) for which we derive a second reciprocity theorem. The wave field in state \(B\) obeys in \(V\) again equation (14). For state \(A\) we define an adjoint medium (or, for quantum mechanics, an adjoint potential), with coefficients \(b_n = (-1)^n a_n^*\), \(b = b^*\), \(d_1 = d_1^*\) and \(d_2 = d_2^*\). Here the bar denotes the adjoint medium and the asterisk denotes complex conjugation. When the original medium is dissipative, the adjoint medium is effectual [22–24] (a wave propagating through an effectual medium gains energy; effectual media are usually associated with a computational state). The wave operator for the adjoint medium is defined as

\[
\mathcal{W} = \mathcal{D} - \sum_{n=0}^{N} (-i\omega)^n a_n^*.
\]

(16)

Note that

\[
\mathcal{W}^* = \mathcal{W}.
\]

(17)

We define \(u_A^*\) as the solution of the wave equation for the adjoint medium, hence

\[
\mathcal{W} u_A^*(x, \omega) = i\omega s_A^*(x, \omega).
\]

(18)

Taking the complex conjugate of both sides of this equation, using equation (17), gives

\[
\mathcal{W} u_A^*(x, \omega) = -i\omega s_A^*(x, \omega).
\]

(19)

We obtain our second reciprocity theorem by substituting \(f = u_A^*\) and \(g = u_B\) into equation (12), using equations (19) and (14). We thus obtain

\[
i\omega \int_V [u_A^* s_B + s_A^* u_B] \, dx = \oint_S J(u_A^*, u_B) \, ds.
\]

(20)
We call this the reciprocity theorem of the correlation type [21, 25], because products like $u_A^* s_B$ in the frequency domain correspond to correlations in the time domain.

A special case is obtained when we consider a lossless medium (which implies we may omit the bars) and take states $A$ and $B$ identical (implying we may also omit the subscripts $A$ and $B$). Equation (20) yields for this situation

$$\frac{1}{4} \int_V [u^* s + s^* u] \, dx = \frac{1}{4 i \omega} \oint_S \mathcal{J}(u^*, u) \, dx.$$  \hspace{1cm} (21)

This expression formulates power conservation, except for the Schrödinger equation, in which case equation (21), multiplied by $4 \omega/h$, stands for conservation of probability.

IV. CLOSED-BOUNDARY GREEN’S FUNCTION REPRESENTATION

A. Reciprocity of the Green’s function

We apply the reciprocity theorem of the convolution type (equation 15) to Green’s functions. For state $A$ we define the Green’s function $G(x, x_A, \omega)$ as the solution of wave equation (13), with the source replaced by a unit point source, $\delta(x - x_A)$, with $x_A$ inside $\mathbb{V}$. Hence

$$WG(x, x_A, \omega) = i \omega \delta(x - x_A).$$  \hspace{1cm} (22)

In a similar way, for state $B$ we define $G(x, x_B, \omega)$ as the response to a unit point source at $x_B$ inside $\mathbb{V}$, hence

$$WG(x, x_B, \omega) = i \omega \delta(x - x_B).$$ \hspace{1cm} (23)

The coefficients of $W$ are the same in both states, inside as well as outside $\mathbb{V}$. As boundary condition we impose the physical radiation condition of outgoing waves at infinity, which corresponds to causality in the time domain, i.e., $G(x, x_A, t) = G(x, x_B, t) = 0$ for $t < 0$. In other words, $G(x, x_A, \omega)$ and $G(x, x_B, \omega)$ are forward propagating Green’s functions. We make the following substitutions in the reciprocity theorem of the convolution type (equation 15)

$$s_A = \delta(x - x_A),$$  \hspace{1cm} (24)

$$u_A = G(x, x_A, \omega),$$  \hspace{1cm} (25)

$$s_B = \delta(x - x_B),$$  \hspace{1cm} (26)

$$u_B = G(x, x_B, \omega).$$ \hspace{1cm} (27)

The boundary integral on the right-hand side is independent of the choice for $\mathcal{S}$ (as long as it encloses $x_A$ and $x_B$). Hence, it vanishes on account of the Sommerfeld radiation condition for outgoing waves at infinity. As a result, equation (15) yields for this situation

$$G(x_B, x_A, \omega) = G(x_A, x_B, \omega).$$ \hspace{1cm} (28)

This expression formulates source-receiver reciprocity.

B. The homogeneous Green’s function

We apply the reciprocity theorem of the correlation type (equation 20) to Green’s functions. For state $A$ we define the Green’s function $G(x, x_A, \omega)$ in the adjoint medium as the solution of

$$WG(x, x_A, \omega) = i \omega \delta(x - x_A),$$  \hspace{1cm} (29)

with $x_A$ in $\mathbb{V}$, and we impose again the condition of outgoing waves at infinity. Taking the complex conjugate of both sides of equation (29) gives

$$WG^*(x, x_A, \omega) = - i \omega \delta(x - x_A).$$ \hspace{1cm} (30)

$G^*(x, x_A, \omega)$ obeys the non-physical radiation condition of incoming waves at infinity, which corresponds to acausality in the time domain, i.e., $G(x, x_A, t) = 0$ for $t > 0$. In other words, $G^*(x, x_A, \omega)$ is a backward propagating Green’s function. For state $B$ we choose again $G(x, x_B, \omega)$, obeying equation (23), with $x_B$ in $\mathbb{V}$. We substitute

$$s_A = \delta(x - x_A),$$  \hspace{1cm} (31)

$$u_A = G(x, x_A, \omega),$$ \hspace{1cm} (32)

for state $A$, and equations (26) and (27) for state $B$, into the reciprocity theorem of the correlation type (equation 20). This gives

$$G(x_A, x_B, \omega) + G^*(x_B, x_A, \omega) = \frac{1}{i \omega} \oint_S \mathcal{J}(\bar{G}^*(x, x_A, \omega), G(x, x_B, \omega)) \, dx.$$ \hspace{1cm} (33)

Using equation (28) this can be rewritten as

$$G_{h_B}(x_B, x_A, \omega) = \frac{1}{i \omega} \oint_S \mathcal{J}(\bar{G}^*(x, x_A, \omega), G(x, x_B, \omega)) \, dx,$$ \hspace{1cm} (34)

with the homogeneous Green’s function $G_{h_B}(x_B, x_A, \omega)$ defined as

$$G_{h_B}(x_B, x_A, \omega) = G(x_B, x_A, \omega) + \bar{G}^*(x_B, x_A, \omega).$$ \hspace{1cm} (35)

By combining equations (22) and (30), it follows that $G_{h_B}(x, x_A, \omega)$ obeys the following wave equation

$$WG_{h_B}(x, x_A, \omega) = 0.$$ \hspace{1cm} (36)

This is a homogeneous differential equation, hence the name “homogeneous Green’s function” for its solution $G_{h_B}(x, x_A, \omega)$.

Equation (34) is akin to the classical homogeneous Green’s function representation [1, 5]. To demonstrate this, we write the integrand in equation (34) in explicit form for the wave phenomena in the first five rows of Table 1. Using $\mathcal{J}_2$, as defined in equation (10), we obtain

$$G_{h_B}(x_B, x_A, \omega) = \frac{1}{i \omega} \oint_S b(x, \omega) \left( \bar{G}^*(x, x_A, \omega) \partial_t G(x, x_B, \omega) \right)$$

$$- \partial_t \bar{G}^*(x, x_A, \omega) G(x, x_B, \omega) \nu_R \, dx.$$ \hspace{1cm} (37)
Next, we take $b$ constant and real-valued, and introduce a modified Green’s function $\mathcal{G}(x, x_A, \omega)$, obeying the wave equation

$$\frac{1}{b} \nabla \mathcal{G}(x, x_A, \omega) = -\delta(x - x_A). \quad (38)$$

Comparing this with equation (22), it follows that

$$\mathcal{G}(x, x_A, \omega) = -\frac{b}{k\omega} G(x, x_A, \omega). \quad (39)$$

Using this relation, equation (37) can be rewritten as

$$\mathcal{G}_h(x_B, x_A, \omega) = \int_S \mathcal{J}(\mathcal{G}^*(x', x, \omega), G(x, x_B, \omega)) n_i \, dx, \quad (40)$$

with

$$\mathcal{G}_h(x_B, x_A, \omega) = \mathcal{G}(x_B, x_A, \omega) - \hat{\mathcal{G}}^*(x_B, x_A, \omega). \quad (41)$$

For the lossless situation we may omit the bars, in which case equation (40) is the classical homogeneous Green’s function representation [1, 5], with $\mathcal{G}_h(x_B, x_A, \omega) = 2i\lambda \{\mathcal{G}(x_B, x_A, \omega)\}$. 

C. Applications

We briefly discuss a number of applications of the closed-boundary homogeneous Green’s function representation (equation 34).

- Holographic imaging

We apply source-receiver reciprocity (equation 28) to two of the three Green’s functions in equation (34), and replace $x_A$ by the variable $x'$. This gives

$$G_b(x', x_B, \omega) = \frac{1}{i\omega} \int_S \mathcal{J}(\mathcal{G}^*(x', x, \omega), G(x, x_B, \omega)) \, dx. \quad (42)$$

The interpretation is as follows, see also Figure 2(a). The coordinate vector $x_B$ denotes the position of a source in $V$. This source may be either a real source, or it may represent a secondary source caused by a scatterer at $x_B$. $G(x, x_B, \omega)$ represents the response to this source, observed by receivers at $x$ on the boundary $S$. The complex-conjugate Green’s function $\mathcal{G}^*(x', x, \omega)$ back-propagates this response from the boundary $S$ to any image point $x'$ in $V$. The integral in equation (42) is taken along all receivers at $x$ on $S$. The homogeneous Green’s function $G_b(x', x_B, \omega)$ at the left-hand side quantifies the properties of the image. Its (finite) value for $x' = x_B$ represents the image amplitude at the position of the source. The behaviour of $G_b(x', x_B, \omega)$ in some region around $x_B$ (indicated by the dashed circle in Figure 2(a)) quantifies the spatial resolution function. For a lossless medium (omitting the bars in equations (42) and (35)), this summarises the essence of holographic imaging methods in optics [1], acoustics [5] and seismology [3]. For a medium with losses, the bar in $\mathcal{G}^*(x', x, \omega)$ denotes that the back propagation is carried out in the adjoint medium [26, 27]. Because of the exponential growth
of this Green’s function with distance, care must be taken when the response at $S$ is contaminated with noise.

- Time-reversed wave propagation

Using $J(f, g) = -J(g, f)$ (see equations (10) and (11)), source-receiver reciprocity (equation 28), replacing $x_B$ by the variable $x$, and assuming the medium is lossless, we obtain from equation (34)

$$-i\omega G_h(x', x, \omega) = \oint_S J(G(x', x, \omega), G^*(x, x, x, \omega))dx.$$  \hspace{1cm} (43)

or, in the time domain,

$$\frac{\partial G_h(x', x, t)}{\partial t} = \oint_S J_r(G(x', x, t), G(x, x, t, -t))dx.$$  \hspace{1cm} (44)

Here the time-domain homogeneous Green’s function is defined as

$$G_h(x', x, t) = G(x', x, t) + G(x, x, t, -t)$$  \hspace{1cm} (45)

and $J(f, g)$ denotes the time-domain equivalent of $J(f, g)$, meaning that products of functions in equations (10) and (11) are replaced by convolutions. The interpretation of equation (44) is as follows, see also Figure 2(b). $G(x, x, t)$ represents the impulse response to a source at $x_A$, observed by receivers at $x$ on the boundary $S$. In a time-reversal experiment, the time-reversed response $G(x, x, t, -t)$ is fed to sources at $x$ on the boundary $S$, which physically emit a wave field into the medium. The propagation of this wave field through the medium to any location $x'$ is described by $G(x', x, t)$. The integral in equation (44) is taken along all sources at $x$ on $S$. The left-hand side of equation (44), with the homogeneous Green’s function defined by equation (45), quantifies the fact that, for $t < 0$, a back propagating field $\partial G(x', x, t, -t)/\partial t$ converges to the focal point $x_A$ and, for $t > 0$, a forward propagating field $\partial G(x', x, t, -t)/\partial t$ propagates from a virtual source at $x_A$ to any observation point $x'$. This summarises the mathematical justification [28, 29] of the principle of time-reversed wave propagation [7, 30–32].

- Green’s function retrieval

Using $J(f, g) = -J(g, f)$, source-receiver reciprocity, and assuming the medium is lossless, we obtain from equation (34)

$$\frac{\partial G_h(x_B, x, t)}{\partial t} = \oint_S J_r(G(x_B, x, t), G(x, x, t, -t))dx.$$  \hspace{1cm} (46)

The interpretation is as follows, see also Figure 2(c). $G(x_A, x, t)$ and $G(x_B, x, t)$ represent the response to a source at $x$ on $S$, observed by two receivers at $x_A$ and $x_B$, respectively. According to equations (10) and (11), the integrand describes a specific combination of convolutions of $G(x_A, x, -t)$ and $G(x_B, x, t)$, which is equivalent to cross correlations of $G(x_A, x, t)$ and $G(x_B, x, t)$. The integration in equation (46) takes place along the sources at $x$ on $S$. According to the left-hand side, this results in the retrieval of the (time-derivative of) the Green’s function and its time-reversal between the receivers $x_A$ and $x_B$. In other words, the receiver at $x_A$ is turned into a virtual source, of which the response is observed by a receiver at $x_B$. This summarises the mathematical justification [11, 33, 34] of the principle of Green’s function retrieval by cross correlation in open systems [10, 12, 35–37].

V. SINGLE-SIDED GREEN’S FUNCTION REPRESENTATION

A. Modified reciprocity theorems

The applications discussed in the previous section all rely on the assumption that the medium (or potential) can be accessed from a closed boundary $S$. In many practical situations the medium can be accessed from one side only. To account for this, we modify the reciprocity theorems of equations (15) and (20). We replace the closed boundary configuration of Figure 1 by that of Figure 3, where $S$ consists of $S_0$, $S_A$, and $S_{cyl}$. Here $S_0$ is the accessible boundary, i.e., the boundary at which measurements can be carried out. This boundary may be horizontal (defined as $x_3 = x_{3,0}$), or curved. $S_A$ is a horizontal boundary below $S_0$, containing $x_A$. Hence, it is defined as $x_3 = x_{3,A}$. Finally, $S_{cyl}$ is, for the 3D situation, a cylindrical boundary with a vertical axis through $x_A$ and infinite radius ($r \to \infty$). This cylindrical boundary exists between $S_0$ and $S_A$ and closes the boundary $S$. For the 2D situation, $S_{cyl}$ consists of two vertical lines between $S_0$ and $S_A$, one at $x_1 \to -\infty$ and one at $x_1 \to +\infty$. The domain enclosed by $S$ is named $V_A$ (the subscript $A$ denoting that this domain depends on the depth of $x_A$). Outside a sphere (or, in 2D, a circle) with finite radius the medium is again homogeneous and lossless in both states.

The contributions of the boundary integrals over $S_{cyl}$ in equations (15) and (20) vanish, but (specifically for equation 20) for another reason than Sommerfeld’s radiation condition. The reasoning is as follows [38].
integrands contain products of functions which each decay with \(1/r\) (in 3D), or \(1/\sqrt{r}\) (in 2D), for \(r \to \infty\). Hence, the integrands decay with \(1/r^2\) (in 3D), or \(1/r\) (in 2D), for \(r \to \infty\). The surface area of \(S_{\text{cy}l}\) is proportional to \(r\) (in 3D) or 1 (in 2D), hence, the integrals decay with \(1/r\) (in 3D and in 2D), and thus vanish for \(r \to \infty\). This implies that we can restrict the integration in the right-hand sides of equations (15) and (20) to the boundaries \(S_{\text{o}}\) and \(S_{\text{A}}\).

For the interaction quantity at the horizontal boundary \(S_{\text{A}}\) (at which \(n_3 = +1\)) we take
\[
\mathcal{J}(f, g) = b[f(\partial_{3}g) - (\partial_{3}f)g].
\]
This strictly holds for the wave phenomena represented by the first five rows in Table 1 (see equation 10), and it holds under the assumption of slowly varying medium parameters for the flexural wave equation, represented by the sixth row in Table 1 (see equations (A18) and (A19) in Appendix A, where \(b\) in the sixth row in Table 1 is defined). For the boundary integrals along \(S_{\text{A}}\) we derive in Appendix B
\[
\mathcal{J}(u_{A}, u_{B})dx = -i\omega\int_{S_{\text{A}}} (u_{A}^{+}u_{B}^{-} - u_{A}^{-}u_{B}^{+})dx,
\]
and
\[
\mathcal{J}(u_{A}^{+}, u_{B})dx = i\omega\int_{S_{\text{A}}} ((u_{A}^{+})^{*}u_{B}^{-} - (u_{A}^{-})^{*}u_{B}^{+})dx.
\]

Here \(u^{+}\) and \(u^{-}\) represent the flux-normalised downgoing (+) and upgoing (−) constituents of the wave field \(u\) at \(S_{\text{A}}\). According to equation (B24), they are related via
\[
u = L_{1}\{u^{+} + u^{-}\},
\]
with composition operator \(L_{1}\) defined in equation (B19). Similarly, the fields in the adjoint medium are at \(S_{\text{A}}\) related via
\[
\tilde{u} = L_{1}\{\tilde{u}^{+} + \tilde{u}^{-}\}.
\]

Using equations (48) and (49) in equations (15) and (20), respectively, yields for the configuration of Figure 3 the following two modified reciprocity theorems
\[
i\omega\int_{V_{\text{A}}} [u_{ASB} - s_{AS}u_{B}]dx = \int_{S_{\text{A}}} \mathcal{J}(u_{A}, u_{B})dx
\]
and
\[
i\omega\int_{S_{\text{A}}} (u_{A}^{+}u_{B}^{-} - u_{A}^{-}u_{B}^{+})dx
\]

FIG. 4: Illustration of the focusing function \(f_{1}(x, x_{A}, \omega)\), defined in a truncated version of the actual medium.

**B. The homogeneous Green’s function**

We use equations (52) and (53) to derive a single-sided representation for the homogeneous Green’s function.

For state \(A\) we introduce a focusing function \(f_{1}(x, x_{A}, \omega)\) in a truncated version of the actual medium, i.e., in a medium which is identical to the actual medium in \(V_{\text{A}}\), but homogeneous and lossless above \(S_{\text{o}}\) and below \(S_{\text{A}}\) (Figure 4). In \(V_{\text{A}}\) it obeys the source-free wave equation
\[
\nabla f_{1}(x, x_{A}, \omega) = 0.
\]

Analogous to equation (50), for \(x\) on \(S_{\text{A}}\) we express the focusing function as a superposition of flux-normalised downgoing and upgoing constituents, according to
\[
f_{1}(x, x_{A}, \omega) = L_{1}(x)\{f_{1}^{+}(x, x_{A}, \omega) + f_{1}^{-}(x, x_{A}, \omega)\}.
\]

The focusing function is incident to the inhomogeneous medium from the homogeneous lossless half-space above \(S_{\text{o}}\), propagates and scatters in the truncated actual medium in \(V_{\text{A}}\), focuses at \(x_{A}\) on \(S_{\text{A}}\), and continues as a downgoing wave field in the homogeneous lossless half-space below \(S_{\text{A}}\). The focusing conditions read [39–41]
\[
[f_{1}^{+}(x, x_{A}, \omega)]_{x_{3}=x_{3,A}} = \delta(x_{H} - x_{H,A})
\]
and
\[
[f_{1}^{-}(x, x_{A}, \omega)]_{x_{3}=x_{3,A}} = 0.
\]

Here \(x_{H}\) denotes the horizontal components of the coordinate vector \(x\), hence, \(x_{H} = (x_{1}, x_{2})\) in the 3D case, and \(x_{H} = x_{1}\) in the 2D case. Similarly, \(x_{H,A}\) denotes
the horizontal components of the coordinate vector \( x_A \).
In practice \( f_1(x, x_A, \omega) \) must be filtered to avoid unstable behaviour for the evanescent wave components. This implies that the delta function in the focusing condition (equation 56) is spatially band-limited.

For state \( B \) we choose again \( G(x, x_B, \omega) \), obeying equation (23) in the actual medium throughout space. Because \( S_A \) (i.e., the lower boundary of \( V_A \)) is determined by the depth of \( x_A \), it follows that \( x_B \) may lie inside or outside \( V_A \), depending on its position relative to \( x_A \). To account for this, we define the characteristic function \( \chi_A \) as

\[
\chi_A(x_B) = \begin{cases} 
1, & \text{for } x_B \text{ in } V_A, \\
\frac{1}{2}, & \text{for } x_B \text{ on } S, \\
0, & \text{for } x_B \text{ outside } S.
\end{cases}
\]  

Analogous to equation (50), for \( x \) on \( S_A \) we express the Green’s function \( G(x, x_B, \omega) \) as a superposition of flux-normalised downgoing and upgoing constituents, according to

\[
G(x, x_B, \omega) = \mathcal{L}_1(x) \{ G^+(x, x_B, \omega) + G^-(x, x_B, \omega) \}.
\]  

We make the following substitutions in the modified reciprocity theorem of the convolution type (equation 52)

\[
s_A = 0, \quad u_A = f_1(x, x_A, \omega), \quad u_B = f_1^*(x, x_B, \omega), \quad s_B = \delta(x - x_B), \quad u_B = G(x, x_B, \omega), \quad u_B = G^+(x, x_B, \omega).
\]

Using focusing conditions (56) and (57), this gives

\[
\chi_A(x_B) \omega f_1(x_B, x_A, \omega) + i \omega G^-(x_A, x_B, \omega) = \int_{S_0} \mathcal{J} \left( f_1(x, x_A, \omega), G(x, x_B, \omega) \right) d\mathbf{x}. \tag{66}
\]

Next, for state \( A \) we define a focusing function \( f_1(x, x_A, \omega) \) in the adjoint of the truncated medium [42]. In \( V_A \) it obeys the source-free wave equation

\[
\mathcal{W} f_1^*(x, x_A, \omega) = 0. \tag{67}
\]

For \( x \) on \( S_A \), \( f_1(x, x_A, \omega) \) consists of downgoing and upgoing constituents, according to

\[
f_1(x, x_A, \omega) = \mathcal{L}_1(x) \{ f_1^*(x, x_A, \omega) + f_1^*(x, x_A, \omega) \}, \tag{68}
\]

with \( \mathcal{L}_1 = \mathcal{L}_1^* \). The focusing conditions for \( f_1(x, x_A, \omega) \) are the same as those for \( f_1(x, x_A, \omega) \), described by equations (56) and (57). We substitute

\[
s_A = 0, \quad u_A = \overline{f_1}(x, x_A, \omega), \quad u_A = \overline{f_1}(x, x_A, \omega), \quad u_A = \overline{f_1}(x, x_A, \omega), \tag{71}
\]

for state \( A \), and equations (63) – (65) for state \( B \), into the modified reciprocity theorem of the correlation type (equation 53). We thus obtain

\[
\chi_A(x_B) \omega f_1^*(x_B, x_A, \omega) - i \omega G^+(x_A, x_B, \omega) = \int_{S_0} \mathcal{J} \left( f_1^*(x, x_A, \omega), G(x, x_B, \omega) \right) d\mathbf{x}. \tag{72}
\]

We obtain two more relations by replacing the medium parameters in both states in equations (66) and (72) by their adjoints, followed by complex conjugating both sides of the resulting equations. We thus obtain

\[
-\chi_A(x_B) \omega f_1(x_B, x_A, \omega) - i \omega \{ G^+(x_A, x_B, \omega) \}^* = \int_{S_0} \mathcal{J} \left( f_1^*(x, x_A, \omega), \tilde{G}^+(x, x_B, \omega) \right) d\mathbf{x} \tag{73}
\]

and

\[
-\chi_A(x_B) \omega f_1(x_B, x_A, \omega) + i \omega \{ G^+(x_A, x_B, \omega) \}^* = \int_{S_0} \mathcal{J} \left( f_1^*(x, x_A, \omega), \tilde{G}^+(x, x_B, \omega) \right) d\mathbf{x}. \tag{74}
\]

Applying \( \mathcal{L}_1(x) \) to both sides of equations (66) and (72) and combining the results, using equations (28) and (59), gives

\[
\chi_A(x_B) \omega F(x_B, x_A, \omega) + i \omega G(x_B, x_A, \omega) = \int_{S_0} \mathcal{J} \left( F(x, x_A, \omega), G(x, x_B, \omega) \right) d\mathbf{x}, \tag{75}
\]

with

\[
F(x, x_A, \omega) = \mathcal{L}_1(x_A) \{ f_1(x, x_A, \omega) - \tilde{f}_1^*(x, x_A, \omega) \}. \tag{76}
\]

Note that on the right-hand side of equation (75) we interchanged the order of the application of operator \( \mathcal{L}_1(x_A) \) and the integration, which is allowed because the operator acts on the coordinate \( x_A \) at \( S_A \), whereas the integration takes place along the coordinate \( x \) at \( S_0 \). Similarly, applying \( \mathcal{L}_1(x_A) \) to both sides of equations (73) and (74) and combining the results, using \( \mathcal{L}_1^* = \mathcal{L}_1 \) and equations (28), (59), and (76), gives

\[
-\chi_A(x_B) \omega F(x_B, x_A, \omega) + i \omega \tilde{G}^+(x_B, x_A, \omega) = \int_{S_0} \mathcal{J} \left( F(x, x_A, \omega), \tilde{G}^+(x, x_B, \omega) \right) d\mathbf{x}. \tag{77}
\]

By combining equations (75) and (77), the focusing functions \( F(x_B, x_A, \omega) \) on the left-hand sides cancel, hence

\[
G_h(x_B, x_A, \omega) = \frac{1}{i \omega} \int_{S_0} \mathcal{J} \left( F(x, x_A, \omega), G_h(x, x_B, \omega) \right) d\mathbf{x}, \tag{78}
\]

where \( G_h(x_B, x_A, \omega) \) is the homogeneous Green’s function, defined in equation (35). Equation (78) is the main result of this paper. It is a representation of the unified homogeneous Green’s function \( G_h(x_B, x_A, \omega) \) between any two points \( x_A \) and \( x_B \) in an inhomogeneous
medium (or potential), expressed in terms of an integral over a single accessible boundary $S_0$. In contrast, the closed-boundary representation for the same unified homogeneous Green’s function (equation (34)) is expressed in terms of a boundary $S$ which encloses the two points $x_A$ and $x_B$.

### C. Applications

We discuss the same applications as in section IV C, but this time using as starting point the single-sided homogeneous Green’s function representation (equation 78). Since each of these applications makes use of the focusing function $F(x,x_A,\omega)$, or, in the time domain, $F(x,x_A,t)$, we first briefly indicate how to obtain this function in practice. For this, we distinguish between model-driven and data-driven approaches. When the medium between $S_0$ and $S_A$ is accurately known (including all scatterers), the transmission response between these boundaries can be numerically modeled (including multiple scattering) and subsequently inverted [41]. This forms the basis for obtaining $F(x,x_A,\omega)$ in a model-driven way. Alternatively, to avoid inversion of the transmission response, the focusing function can be modeled directly, following a recursive Kirchhoff-Helmholtz wavefield extrapolation approach [43, 44], starting with the focused field at $S_A$ and recursively moving upward. These model-driven approaches hold for media with or without losses. When only a smooth background medium is known, the focusing function (including multiple scattering) can be retrieved from reflection measurements at the boundary $S_0$, using a 2D or 3D version of the single-sided Marchenko method [39]. This data-driven approach holds for media without losses. It can be extended to media with losses [42], but this requires measurements at $S_0$ and $S_A$.

#### Holographic imaging

We apply source-receiver reciprocity to the Green’s function at the left-hand side of (78), and replace $x_A$ by the variable $x'$. This gives

$$G_h(x',x_B,\omega) = \frac{1}{i\omega} \int_{S_0} J(F(x,x',\omega),G_h(x,x_B,\omega))dx,$$

(79)

see Figure 5(a). The main difference with the holographic imaging method, described by equation (42), is that the closed integration boundary $S$ has been replaced by the single (open) boundary $S_0$ and that the back-propagating Green’s function $G^*(x',x,\omega)$ has been replaced by the focusing function $F(x,x',\omega)$. Moreover, the response to which the focusing function is applied is the homogeneous Green’s function $G_h(x,x_B,\omega)$, instead of $G(x,x_B,\omega)$ in equation (42). The integral in equation (79) is taken along all receivers at $x$ on $S_0$. The homogeneous Green’s function on the left-hand side is the same as that in equation (42), hence, it quantifies the spatial resolution function in some region around $x_B$ (indicated by the dashed circle in Figure 5(a)).

#### “Time-reversed” wave propagation

In the time-reversal approach discussed in section IV C, a time-reversed field is physically emitted into the medium. For the single-sided version we cannot make use of equation (78), because the homogeneous Green’s function $G_h(x,x_B,\omega)$ in the integrand is not physical. Therefore we return to equation (75), in which the integrand contains the physical (i.e., causal) Green’s function $G(x,x_B,\omega)$. Using $J(f,g) = -\int g(f)J$, source-receiver reciprocity, replacing $x_B$ by the variable $x'$, and assuming the medium is lossless, we obtain from equation (75)

$$\frac{\partial G(x',x_A,t)}{\partial t} + \chi_A(x') \frac{\partial F(x',x_A,t)}{\partial t} = \int_{S_0} J_r(G'(x',t),F(x,x_A,t))dx,$$

(80)
see Figure 5(b). In comparison with the time-reversal method, described by equation (44), the focusing function \( F(\mathbf{x}, \mathbf{x}_A, t) \) instead of \( G(\mathbf{x}, \mathbf{x}_A, -t) \) is fed to the sources at \( \mathbf{x} \) on \( S_0 \) (instead of \( S \)) and physically emitted into the medium. The integral in equation (80) is taken along all sources at \( \mathbf{x} \) on \( S_0 \). The left-hand side shows that, for \( \mathbf{x}' \) below \( \mathbf{x}_A \) (where \( \chi_A(\mathbf{x}') = 0 \)), the field \( \partial G(\mathbf{x}', \mathbf{x}_A, t)/\partial t \) propagates from a virtual source at \( \mathbf{x}_A \) to any observation point \( \mathbf{x}' \). For \( \mathbf{x}' \) above \( \mathbf{x}_A \) the virtual-source field is contaminated by the focusing function \( \partial F(\mathbf{x}', \mathbf{x}_A, t)/\partial t \). Note, however, that only the direct arrival of the Green’s function overlaps with the focusing function [41]. Beyond the direct arrival time the focusing function is zero and hence does not contaminate the virtual-source field \( \partial G(\mathbf{x}', \mathbf{x}_A, t)/\partial t \).

### Green’s function retrieval

Using \( J(f, g) = -J(g, f) \), source-receiver reciprocity, and assuming the medium is lossless, we obtain from equation (78)

\[
\frac{\partial G_b(\mathbf{x}_B, \mathbf{x}_A, t)}{\partial t} = \int_{S_0} J_t(G_b(\mathbf{x}_B, \mathbf{x}, t), F(\mathbf{x}, \mathbf{x}_A, t))d\mathbf{x},
\]

(81)

see Figure 5(c). Instead of cross correlating the Green’s functions \( G(\mathbf{x}_A, \mathbf{x}, t) \) and \( G(\mathbf{x}_B, \mathbf{x}, t) \), as in equation (46), here the homogeneous Green’s function \( G_b(\mathbf{x}_B, \mathbf{x}, t) \), observed by a receiver at \( \mathbf{x}_B \), is convolved with the focusing function \( F(\mathbf{x}, \mathbf{x}_A, t) \), focused at a receiver at \( \mathbf{x}_A \). The integration takes place along the sources at \( \mathbf{x} \) on \( S_0 \) (instead of \( S \)). The left-hand side is the same as in equation (46), i.e., the response to a virtual source at \( \mathbf{x}_A \), observed by a receiver at \( \mathbf{x}_B \). Note that, since the focusing function \( F(\mathbf{x}, \mathbf{x}_A, t) \) is related to the inverse of the transmission response between \( S_0 \) and \( S_A \), Green’s function retrieval using equation (81) is akin to a method called “Green’s function retrieval by multidimensional deconvolution” [45, 46], in which the Green’s function \( G(\mathbf{x}_A, \mathbf{x}, t) \) is inverted rather than reversed in time.

### VI. CONCLUSIONS

We have derived a representation for the homogeneous Green’s function of a unified scalar wave equation. Unlike the classical representation, which involves an integral along a closed boundary, this representation is expressed as an integral over a single boundary only. It accounts for multiple scattering and holds in media with losses. This representation is particularly useful when the medium of interest is accessible from one side only. Applications are found in holographic imaging methods, time-reversed wave propagation and Green’s function retrieval.

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### Appendix A: Flexural waves

Consider a 2D inhomogeneous isotropic thin plate in the \((x_1, x_3)\)-plane. The equation of motion reads [47, 48]

\[
\partial_t Q - \rho \frac{\partial^2 u_2}{\partial t^2} = -f_2,
\]

(A1)

where

\[
Q_i = \partial_j M_{ij},
\]

(A2)

with \( u_2(\mathbf{x}, t) \) the transverse particle displacement, \( f_2(\mathbf{x}, t) \) the area density of transverse external force, \( Q_i(\mathbf{x}, t) \) the shear stress, \( M_{ij}(\mathbf{x}, t) \) the moment and \( \rho(\mathbf{x}) \) the area density of mass. The subscripts \( i \) and \( j \) take on the values 1 and 3 only. Einstein’s summation convention applies to repeated subscripts. For an isotropic plate, the moments are defined as [47, 48]

\[
M_{11} = -[\partial_t \partial_i \partial_j + \nu d \delta_{ij} d]u_2, \quad M_{33} = -(\nu d \partial_t \partial_1 + d \partial_t \partial_3)u_2, \quad M_{13} = M_{31} = -(1 - \nu) d \partial_t \delta_{13}u_2,
\]

(A3)\(\)\(\)\(\)\(\)

with \( d(\mathbf{x}) \) the bending stiffness and \( \nu(\mathbf{x}) \) the Poisson ratio. These equations can be captured by the single equation

\[
M_{ij} = -[(1 - \nu) d \partial_t \partial_i + \delta_{ij} \nu d \partial_t \partial_k]u_2,
\]

(A6)

where \( \delta_{ij} \) is the Kronecker delta function. Substitution of equations (A2) and (A6) into equation (A1) gives the flexural wave equation for the transverse particle velocity \( v_2(\mathbf{x}, t) = \partial u_2(\mathbf{x}, t)/\partial t \), hence,

\[
-[\partial_t \partial_j d_1 \partial_i \partial_j + \partial_i \partial_t d_2 \partial_j \partial_j]v_2 - \rho \frac{\partial^2 v_2}{\partial t^2} = -\frac{\partial f_2}{\partial t},
\]

(A7)

with

\[
d_1 = (1 - \nu)d, \quad d_2 = \nu d.
\]

(A8)\(\)\(\)

Equation (A7), transformed to the frequency domain, reads

\[
-\partial_j \partial_i d_1 \partial_t \partial_j + \partial_i \partial_t d_2 \partial_j \partial_j]v_2 + \rho \omega^2 v_2 = i \omega f_2,
\]

(A10)

with \( v_2(\mathbf{x}, \omega) \) and \( f_2(\mathbf{x}, \omega) \) being the Fourier transforms of \( v_2(\mathbf{x}, t) \) and \( f_2(\mathbf{x}, t) \) in equation (A7). Equation (A10) is covered by the unified wave equation in the frequency domain (equations (3) and (4)), with the quantities \( u, s \) and \( a_2 \) given in the sixth row of Table 1 and the differential operator \( D \) defined in equation (6). To account for general loss mechanisms, we assume that
\( \rho(x, \omega), d_1(x, \omega) \) and \( d_2(x, \omega) \) may be complex-valued and frequency-dependent. This exact wave equation underlies the closed-boundary representation of the homogeneous Green’s function, equation (34), for the situation of flexural waves.

For the derivation of the single-sided representation we assume that the plate is only weakly inhomogeneous, so that the spatial derivatives of the coefficients \( \rho(x, \omega) \), \( d_1(x, \omega) \) and \( d_2(x, \omega) \) are small in comparison with those of the wave field \( v_2(x, \omega) \). With this assumption, equation (A10) simplifies to

\[
-d\partial_i \partial_j \partial_j v_2 + \rho \omega^2 v_2 = \omega f_2. \quad (A11)
\]

We define operator \( \mathcal{D}_4 \) as

\[
\mathcal{D}_4 = -d\partial_i \partial_j \partial_j. \quad (A12)
\]

For \( J_4(f, g) \), obeying equation (9), we thus obtain

\[
J_4(f, g) = d[\partial_i \partial_j f \partial_j g - f \partial_i \partial_j \partial_j g] + \partial_i f \partial_i \partial_j g - (\partial_i f \partial_j g) n_i. \quad (A13)
\]

This expression can be further simplified. To this end we first rewrite equation (A11) as follows

\[
(\partial_i - k_i^2)(\partial_i \partial_j + k_j^2) v_2 = -i\omega f_2/d, \quad (A14)
\]

with the flexural wavenumber \( k_i^2 \) defined as

\[
k_i^2 = \omega \sqrt{\frac{\rho}{d}}. \quad (A15)
\]

In any source-free region, equation (A14) decouples into the following two equations

\[
(\partial_i - k_i^2) v_2 = 0, \quad (A16)
\]

\[
(\partial_i \partial_j + k_j^2) v_2 = 0. \quad (A17)
\]

Equation (A16) accounts for damped solutions of equation (A14), whereas equation (A17) accounts for waves. Assuming that \( S \) (where \( J_4(f, g) \) is defined) is source-free and that \( f \) and \( g \) both obey wave equation (A17), equation (A13) simplifies to

\[
J_4(f, g) = 2k_i^2 d[\partial_i f \partial_i g - (\partial_i f) g] n_i. \quad (A18)
\]

Note that this expression is the same as that for \( J_2(f, g) \), given in equation (10), if we define

\[
b = 2k_i^2 d = 2\omega \sqrt{\rho d}. \quad (A19)
\]

**Appendix B: Decomposition**

Our aim is to derive decomposed versions of the boundary integrals in reciprocity theorems (15) and (20) for the part \( S_A \) of the closed boundary \( S \). Hence, we seek decomposed versions of

\[
\int_{S_A} J(u_A, u_B) dx \quad \text{and} \quad \int_{S_A} J(u_A^*, u_B) dx. \quad (B1)
\]

Since \( S_A \) is a horizontal boundary, its normal vector is defined as \( n = (n_1, n_2, n_3) = (0, 0, 1) \) for 3D situations, and \( n = (n_1, n_3) = (0, 1) \) for 2D situations. In the following we define the interaction quantity at \( S_A \) as

\[
J(f, g) = b f(\partial_3 g) - (\partial_3 f) g, \quad (B2)
\]

which strictly holds for the wave phenomena represented by the first five rows in Table 1 (see equation 10), and which holds under the assumption of slowly varying medium parameters for the flexural wave equation, represented by the sixth row in Table 1 (see equations (A18) and (A19)).

According to equation (B2), we need expressions for the first order derivative \( \partial_3 \), which we derive from expressions for second order derivatives. For the wave phenomena represented by the first five rows in Table 1, we find from equations (3), (4) and (5), assuming the medium is source-free at \( S_A \),

\[
\partial_3(b \partial_3 u) = -K_2 u, \quad (B3)
\]

with

\[
K_2 = -\sum_{n=0}^{N} (-i \omega)^n \alpha_n + \partial_n b \partial_n. \quad (B4)
\]

For the 3D situation, Greek subscripts take on the values 1 and 2, and the summation convention holds for repeated subscripts. For the 2D situation, Greek subscripts take on the value 1 only. Subscript 2 in operator \( K_2 \) denotes that this operator accounts for second order differentiation in the \( x_3 \)-direction.

For flexural waves, represented by the sixth row in Table 1, we assume that the plate is weakly inhomogeneous, so that the spatial derivatives of the coefficients are small in comparison with those of the wave field. Under these assumptions (and the assumption that \( S_A \) is source-free), equations (A15), (A17) and (A19) yield again equations (B3) and (B4), but this time with \( \alpha_0 = \alpha_1 = 0 \) and \( \alpha_2 = 2\alpha_2 = 2\rho \). We define an operator \( \mathcal{H}_2 \) as

\[
\mathcal{H}_2 = b^{-\frac{1}{2}} K_2 b^{-\frac{1}{2}} \quad (B5)
\]

\[
= -\sum_{n=0}^{N} \alpha_n (-i \omega)^n + \partial_n \partial_n,
\]

with \([38, 49]\)

\[
\alpha_n' = \alpha_n/b \quad \text{for} \quad n > 0. \quad (B6)
\]

Note that \( \partial_3 b \partial_3 \) in operator \( \mathcal{K}_2 \) (equation B4) has been simplified into \( \partial_3 \partial_3 \) in operator \( \mathcal{H}_2 \) (equation B5). Using this definition of \( \mathcal{H}_2 \), we rewrite equation (B3) as

\[
\partial_3(b \partial_3 u) = -b^{-\frac{1}{2}} \mathcal{H}_2(b^{-\frac{1}{2}} u). \quad (B8)
\]
We define an operator $\mathcal{H}_1$ as the square root of $\mathcal{H}_2$, according to \cite{50-52}
\begin{equation}
\mathcal{H}_1 \mathcal{H}_1 = \mathcal{H}_2 \quad \text{or} \quad \mathcal{H}_1 = \mathcal{H}_2^{1/2}.
\end{equation}

Note that $\mathcal{H}_1$ is a pseudo-differential operator \cite{24, 53-59}. The square root of an operator is not uniquely defined. We take the square root such that the imaginary part of the eigenvalue spectrum of $\mathcal{H}_1$ has the same sign as that of $\mathcal{H}_2$, i.e., positive for a dissipative medium and negative for an effectual medium (for details see \cite{24}, keeping in mind that the imaginary unit $j$ in that paper has been replaced by $-i$ in the present paper, to be consistent with the use of $i$ in Schrödinger’s equation).

We combine equation (B8) and the trivial equation $\partial_{\mathbf{r}} \mathbf{q} = \mathbf{A} \mathbf{q}$, according to
\begin{equation}
\partial_{\mathbf{r}} \mathbf{q} = \mathbf{A} \mathbf{q},
\end{equation}
with
\begin{equation}
\mathbf{A} = \begin{pmatrix}
0 & i \omega \mathbf{H}_2 b^2 z_1 \\
-\frac{i}{\omega} b^2 & 0
\end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix}
b u \\
\mathbf{i} \mathbf{D} \mathbf{u}
\end{pmatrix}.
\end{equation}

With this definition of $\mathbf{q}$, the integrals in equation (B1) (with $\mathcal{J}$ defined in equation B2) can be written as
\begin{equation}
\int_{S_A} \mathcal{J}(u_A, u_B) dx = i \omega \int_{S_A} \mathbf{q}^t \mathbf{N} \mathbf{q} dx, \quad (B12)
\end{equation}
\begin{equation}
\int_{S_A} \mathcal{J}(\bar{u}_A, u_B) dx = i \omega \int_{S_A} \mathbf{q}^t \mathbf{K} \mathbf{q} dx, \quad (B13)
\end{equation}

with
\begin{equation}
\mathbf{N} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \quad \text{and} \quad \mathbf{K} = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\end{equation}

In equation (B12), $\mathbf{q}^t$ denotes the transposed of $\mathbf{q}$.\textsuperscript{A}, whereas $\mathbf{q}^t$ in equation (B13) denotes the adjoint (i.e., the complex conjugate transposed) of $\mathbf{q}$.\textsuperscript{A}

We decompose matrix $\mathbf{A}$ as follows
\begin{equation}
\mathbf{A} = \mathcal{L} \mathcal{H} \mathcal{L}^{-1},
\end{equation}
with
\begin{equation}
\mathcal{L} = \begin{pmatrix}
\mathcal{L}_1 & \mathcal{L}_2 \\
\mathcal{L}_2 & -\mathcal{L}_1
\end{pmatrix},
\end{equation}
\begin{equation}
\mathcal{H} = \begin{pmatrix}
ni \mathcal{H}_1 & 0 \\
0 & -ni \mathcal{H}_1
\end{pmatrix},
\end{equation}
\begin{equation}
\mathcal{L}^{-1} = \frac{1}{2} \begin{pmatrix}
\mathcal{L}^{-1}_1 & \mathcal{L}^{-1}_2 \\
\mathcal{L}^{-1}_2 & -\mathcal{L}^{-1}_1
\end{pmatrix}.
\end{equation}

This decomposition is not unique. We choose
\begin{equation}
\mathcal{L}_1 = (\omega/2)^{1/2} b^{-\frac{1}{2}} \mathcal{H}_1^{1/2},
\end{equation}
\begin{equation}
\mathcal{L}_2 = (\omega)^{-\frac{1}{2}} b^2 \mathcal{H}_1^{1/2},
\end{equation}
\begin{equation}
\mathcal{L}^{-1}_1 = (\omega/2)^{-1/2} \mathcal{H}_1^{1/2} b^2,
\end{equation}
\begin{equation}
\mathcal{L}^{-1}_2 = (\omega)^{-1/2} \mathcal{H}_1^{1/2} b^{-2}.
\end{equation}
Here $\mathcal{H}_1^{1/2}$ is the square root of the square root operator $\mathcal{H}_1$ (with the same choices for the sign of the imaginary part of the eigenvalue spectrum).

We introduce a decomposed field vector $\mathbf{p}$ via
\begin{equation}
\mathbf{q} = \mathcal{L} \mathbf{p}, \quad \text{with} \quad \mathbf{p} = \begin{pmatrix}
u^+ \\
u^-
\end{pmatrix},
\end{equation}
or
\begin{equation}
u = \mathcal{L}_1 \{ \nu^+ + \nu^- \},
\end{equation}
\begin{equation}
\frac{b}{i \omega} \partial_{\mathbf{r}} \nu = \mathcal{L}_2 \{ \nu^+ - \nu^- \}.
\end{equation}

We interpret $\nu^+$ and $\nu^-$ as downgoing (+) and upgoing (−) wave fields \cite{24, 55-59}. With the specific choice for the operators $\mathcal{L}_1$ and $\mathcal{L}_2$, made in equations (B19) and (B20), $\nu^+$ and $\nu^-$ are so-called flux-normalised downgoing and upgoing wave fields. This is explained at the end of this appendix.

Substitution of equation (B23) into equations (B12) and (B13) gives
\begin{equation}
\int_{S_A} \mathcal{J}(u_A, u_B) dx = i \omega \int_{S_A} (\mathbf{L} \mathbf{p})^t \mathbf{N} \mathbf{L} \mathbf{p} dx, \quad (B26)
\end{equation}
\begin{equation}
\int_{S_A} \mathcal{J}(\bar{u}_A, u_B) dx = i \omega \int_{S_A} (\mathbf{L} \mathbf{p})^t \mathbf{K} \mathbf{L} \mathbf{p} dx. \quad (B27)
\end{equation}

We introduce transposed and adjoint operators via
\begin{equation}
\int_{S_A} (\mathbf{U}^t)^t \mathbf{g} dx = \int_{S_A} \mathbf{f}^t \mathbf{U}^t \mathbf{g} dx, \quad (B28)
\end{equation}
\begin{equation}
\int_{S_A} (\mathbf{U}^t)^* \mathbf{g} dx = \int_{S_A} \mathbf{f}^t (\mathbf{U}^t)^* \mathbf{g} dx. \quad (B29)
\end{equation}

Here $\mathbf{U}$ is a (pseudo-)differential operator matrix containing the spatial differential operator $\partial_{\mathbf{r}}$, whereas $\mathbf{U}^t$ and $\mathbf{U}^*$ are the transposed and adjoint (complex conjugate transposed) of $\mathbf{U}$. Furthermore, $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ are vector functions with “sufficient decay” at infinity. Using equations (B28) and (B29), we can rewrite equations (B26) and (B27) as follows
\begin{equation}
\int_{S_A} \mathcal{J}(u_A, u_B) dx = i \omega \int_{S_A} \mathbf{p}^t \mathbf{L}^t \mathbf{N} \mathbf{L} \mathbf{p} dx, \quad (B30)
\end{equation}
\begin{equation}
\int_{S_A} \mathcal{J}(\bar{u}_A, u_B) dx = i \omega \int_{S_A} \mathbf{p}^t \mathbf{L}^t \mathbf{K} \mathbf{L} \mathbf{p} dx. \quad (B31)
\end{equation}

To simplify this result further, we first consider the transposed and adjoint versions of the scalar operators $\mathcal{H}_1$, $\mathcal{L}_1$ and $\mathcal{L}_2$, which are given by \cite{24, 58}
\begin{equation}
\mathcal{H}^t = \mathcal{H}_1^t = \mathcal{H}_1, \quad (B32)
\end{equation}
\begin{equation}
\mathcal{L}_1^t = \mathcal{L}_1^{-1} = \frac{1}{2} \mathcal{L}^{-1}_2, \quad (B33)
\end{equation}
\begin{equation}
\mathcal{L}_2^t = \mathcal{L}_2^{-1} = \frac{1}{2} \mathcal{L}^{-1}_1. \quad (B34)\end{equation}
Using equations (B14), (B16), (B18), (B32), (B33) and (B34), it follows that
\[
\mathcal{L}' \mathcal{N} \mathcal{L} = -\mathcal{N}, \quad (B35)
\]
\[
\mathcal{L}' \mathcal{K} \mathcal{L} = \mathcal{J}, \quad (B36)
\]
with
\[
\mathcal{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (B37)
\]

Using this in equations (B30) and (B31) yields
\[
\int_{S_A} \mathcal{J}(u_A, u_B) \, dx = -i \omega \int_{S_A} p_A' N p_B \, dx, \quad (B38)
\]
\[
\int_{S_A} \mathcal{J}(\bar{u}'_A, u_B) \, dx = i \omega \int_{S_A} p_A' J p_B \, dx, \quad (B39)
\]
or, using equations (B14), (B23) and (B37),
\[
\int_{S_A} \mathcal{J}(u_A, u_B) \, dx = -i \omega \int_{S_A} (u_A^+ u_B - u_A^- u_B^+) \, dx, \quad (B40)
\]
\[
\int_{S_A} \mathcal{J}(\bar{u}'_A, u_B) \, dx = i \omega \int_{S_A} ((\bar{u}'_A)^+ u_B^+ - (\bar{u}'_A)^- u_B^-) \, dx. \quad (B41)
\]

According to equation (21), the left-hand side represents the power flux through \( S_A \). Hence, \( u^+ \) and \( u^- \) on the right-hand side are (power-)flux-normalised downgoing and upgoing wave fields.


J. F. Claerbout, Geophysics 36, 467 (1971).


